2.3 Families of Functions, Transformations, and Symmetry

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Definition

If the graph of any function has origin symmetry, then for any point \((x,y)\) on the graph, there is also a point \((-x,-y)\) on the graph.

Figure: The graph of \(f(x) = x^3\) is an example of a function that has origin symmetry.
Definition

If the graph of any function has \textit{y-axis symmetry}, then for every point \((x,y)\), there is also a point \((-x,y)\) on the graph.

Figure: \(f(x) = x^2\) is an example of a function that has \textit{y-axis symmetry}. 
Definition

The graph of a relation has **x-axis symmetry** if for every point \((x, y)\) on the graph, the point \((x, -y)\) is also on the graph.

Figure: \(x = y^2\) is an example of a relation that has \(y\)-axis symmetry.

Can a function have \(x\)-axis symmetry?
Example 1: Does \( x = y^2 - 1 \) have y-axis symmetry, origin symmetry, x-axis symmetry, or no symmetry?

**Counterexample:** \((x, y) = (3, 2)\) is a solution to the equation \( x = y^2 - 1 \) since \( 3 = 2^2 - 1 \). If the graph of \( x = y^2 - 1 \) had origin symmetry, then \((-3, -2)\) would be a solution to \( x = y^2 - 1 \), but it is not since \(-3 \neq (-2)^2 - 1\). So the graph of the relation cannot have origin symmetry.

![Graph of \( y = x^3 \)](image)

**Figure:** The graph of \( f(x) = x^3 \) is an example of a function that has origin symmetry.
Example 1: Does \( x = y^2 - 1 \) have y-axis symmetry, origin symmetry, x-axis symmetry, or no symmetry?

**Counterexample:** We know that \((x, y) = (3,2)\) is a solution to the equation \( x = y^2 - 1 \). If the graph of \( x = y^2 - 1 \) had y-axis symmetry, then \((-3,2)\) would be a solution to \( x = y^2 - 1 \), but it is not since \(-3 \neq (2)^2 - 1\). So the graph cannot have y-axis symmetry.

**Figure:** \( f(x) = x^2 \) is an example of a function that has y-axis symmetry.
Example: Does $x = y^2 - 1$ have y-axis symmetry, origin symmetry, x-axis symmetry, or no symmetry?

**Definition**

The graph of a relation has \textit{x-axis symmetry} if for every point $(x, y)$ on the graph of the relation, the point $(x, -y)$ is also on the graph.
Example: Does \( x = y^2 - 1 \) have y-axis symmetry, origin symmetry, x-axis symmetry, or no symmetry? Give me an algebraic proof.

**Proof:** To algebraically prove that \( x = y^2 - 1 \) has x-axis symmetry, we need to show that for every point \((x, y)\) on the graph of \( x = y^2 - 1 \), the point \((x, -y)\) is also on the graph. Suppose \( a \) and \( b \) are any real numbers such that \((a, b)\) is a solution to the equation \( x = y^2 - 1 \). Substitution of \((a, b)\) into the equation \( x = y^2 - 1 \) leads to the result:

\[
a = b^2 - 1 \quad \text{or} \quad a + 1 = b^2 \quad \text{or} \quad b = \pm \sqrt{a + 1}
\]

This means that for every point on the graph \((a, \sqrt{a + 1})\), there is a complimentary point \((a, -\sqrt{a + 1})\) also on the graph. Therefore the graph of \( x = y^2 - 1 \) has x-axis symmetry, by definition.
Even and Odd Functions and Functions that are Neither

**Definition**

A function $f(x)$ can be classified as (one of the following):

1. Even
2. Odd
3. Neither Even Nor Odd

**Figure:** A function that is neither: $f(x) = x(x - 2)^2$
Definition

A function is **EVEN** if its graph has y axis symmetry. If substitution of \(-x\) for \(x\) leads to the same equation, i.e., \(f(-x) = f(x)\), then \(f\) is an even function.

Example: An Even Function  
Determine if the function \(f(x) = x^2 - 3\) is an even function.

1. First replace the \(x\) in \(f(x)\) with \(-x\).

\[f(-x) = (-x)^2 - 3 = x^2 - 3\]

2. Now simplify \(f(-x)\)

\[f(-x) = (-x)^2 - 3 = x^2 - 3 = f(x)\]

3. Result: Since \(f(-x) = f(x)\), the given function, \(f\), is an even function, which means it has y-axis symmetry. (note: read the definition of an even function again.)
**Definition**

A function is **ODD** if its graph has origin symmetry. If substitution of \(-x\) for \(x\) leads to the negative version of \(f\), i.e., \(f(-x) = -f(x)\), then \(f\) is an odd function.

**Example: An Odd Function**  
Consider the function \(f(x) = 4x^3 - x\). Test to see if \(f\) is an odd function.

1. First replace the \(x\) in \(f(x)\) with \(-x\).

\[
f(-x) = 4(-x)^3 - (-x)
\]

2. Now simplify \(f(-x)\)

\[
f(-x) = 4(-x)^3 - (-x) = -4x^3 + x = -1(4x^3 - x) = -f(x)
\]

3. Result: Since \(f(-x) = -f(x)\), the given function, \(f\), is an odd function, and it has origin symmetry.
Example: Of a function that is neither Even nor Odd
Function Test the function $f(x) = -x^7 - 3$ for $x$ or $y$ axis symmetry.

1. First we will test to see if the function is even:

   $f(-x) = -(-x)^7 - 3 = -(-x^7) - 3 = x^7 - 3 \neq f(x)$

   Therefore since $f(-x) \neq f(x)$, it follows that $f(x)$ is NOT an even function.

2. Now we will test to see if the function is odd:

   $f(-x) = -(-x)^7 - 3 = -(-x^7) - 3 = x^7 - 3 \neq -f(x)$

   Therefore since $f(-x) \neq -f(x)$, it follows that $f(x)$ is NOT an odd function.

Hence the function $f(x)$ is neither even nor odd. Moreover we can conclude that the graph of $f$ does not have any symmetry.
Definition

If $a, c$, and $d$ are real numbers with $a \neq 0$, then $y = a \cdot f(x - c) + d$ is a **transformation** of the **parent** function $y = f(x)$.

Figure: The graph of $y = \sqrt[3]{x}$

All of the transformations of a function **form a family of functions**. For example, any function of the form $y = a\sqrt[3]{x - c} + d$ is in the cube root family of functions (with **parent** function $f(x) = \sqrt[3]{x}$).
Vertical Shifts of Graphs

**Theorem**

Suppose \( d \) is a positive number and \( f(x) \) is a function. Then

- The graph of \( y = f(x) + d \) is the graph of \( y = f(x) \) shifted vertically upward \( d \) units.

![Diagram of vertical shifts of graphs](image)

**Figure:** The graphs of \( f(x) = x(x - 2)^2 \) and \( f(x) + d \) are given in the left panel; and the graphs of \( f(x) = x^2 \) and \( g(x) = x^2 + 2 \) are presented in the right panel above.
**Theorem**

Suppose $d$ is a positive number and $f(x)$ is a function. Then

- The graph of $y = f(x) - d$ is the graph of $y = f(x)$ shifted vertically downward $d$ units.

**Figure:** The graphs of $f(x) = x(x - 2)^2$ and $f(x) - d$ are given in the left panel; and the graphs of $f(x) = x^2$ and $g(x) = x^2 - 2$ are presented in the right panel above.
Theorem

Let $c$ be a positive number. Then

- The graph of $y = f(x + c)$ is the graph of $y = f(x)$ shifted to the left $c$ units.

Note that this is counterintuitive.

Figure: The graphs of $f(x) = x(x - 2)^2$ and $f(x + c)$ are given in the left panel; and the graphs of $f(x) = x^2$ and $g(x) = (x + 2)^2$ are presented in the right panel above.
**Theorem**

Let $c$ be a positive real number. Then

- The graph of $y = f(x - c)$ is the graph of $y = f(x)$ shifted to the right $c$ units.

Note that this is counterintuitive.

**Figure:** The graphs of $f(x) = x(x - 2)^2$ and $f(x + c)$ are given in the left panel; and the graphs of $f(x) = x^2$ and $g(x) = (x - 2)^2$ are presented in the right panel above.
Theorem

The graph of \( g(x) = -f(x) \) is reflection of \( f(x) \) about the x-axis.

Figure: Graphs of \( f(x) = x^2 \) and \( g(x) = -x^2 \).
Magnification

Definition
Suppose \( a \) is a positive number. The graph of \( g(x) = af(x) \) is called a **magnification** of \( f(x) \).

Theorem
The magnification of \( y = a \cdot f(x) \) is expanded horizontally whenever \( 0 < a < 1 \) and compressed horizontally whenever \( a > 1 \).

Figure: (left) \( y = ax^2 \) and \( 0 < a < 1 \) (right) \( y = ax^2 \) and \( a \geq 1 \).
Reflection and Magnification

Figure: $y = ax^2$, $a \leq -1$, and $y = ax^2$, $-1 < a < 0$
Consider the function $g$ defined by

$$g(x) = a \cdot f(x - c) + d$$

where $a$, $c$, and $d$ are real numbers.

Then

1. $g(x)$ is the “generalized” child graph of parent graph $f(x)$.
2. $c$ represents the horizontal translation of $f$.
3. $a$ represents the reflection/magnification of $f$.
4. $d$ represents the vertical translation of $f$. 

Definition (Multiple Transformations Graphing Algorithm)

Consider the function $g$ defined by

$$g(x) = a \cdot f(x - c) + d$$

where $a$, $c$, and $d$ are real numbers.

In order to graph $g(x)$ it is recommended to take the following steps:

1. Identify and graph the parent graph $f(x)$, of $g(x)$. Select two or three points on the parent graph to shift (follow the evolution of) in steps 2-4.

2. (c) Translate (or shift) the selected points $c$ units horizontally, i.e. apply $f(x \pm c)$.

3. (a) Reflect/magnify the points in the previous graph by multiplying each $y$-coordinate by “$a$.”

4. (d) Translate (or shift) the points in the previous graph vertically $d$ units.

Note: If you are asked to graph, for example, $f(x) = -3 \sqrt[3]{x - 1} + 1$, then you should rename $f(x)$ and give it the new name of $g(x)$. Then find $g$’s parent graph (it’s $f(x) = \sqrt[3]{x}$ for this example).
Classroom Examples  Use the theory on translations to graph

1. $g(x) = -2\sqrt{x - 1} + 1$

2. $g(x) = -2|x + 2| - 3$

3. $g(x) = (x - 1)^2 - 2$.

Then use the graph to find the solution set to $g(x) \geq 0$.

After graphing each function, state the domain and range of the function, intervals for which the function is increasing or decreasing, and a pair of limit statements describing the end behavior of each graph.
Use translations to graph: \( g(x) = -3 \cdot 3^{\sqrt{x - 1}} + 1 \)

**Step 1:** identify and graph the parent function: \( f(x) = 3^{\sqrt{x}} \). We select two or three points on the parent graph to shift (follow the evolution of) in steps 2-4. Select, for example, the points \((-8, -2), (0, 0), \) and \((8, 2)\).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>-8</td>
<td>-2</td>
</tr>
</tbody>
</table>

**Step 2:** graph: \( g_1(x) = 3^{\sqrt{x - 1}} = f(x - 1) \). Shift the selected points horizontally \( c = 1 \) unit right. (Alternatively, add one to each \( x \) value in the table, and \( y \) values stay the same.)
Use translations to graph: \( g(x) = -3 \cdot 3^{\sqrt[3]{x-1}} + 1 \)

**Step 2:** graph: \( g_1(x) = 3^{\sqrt[3]{x-1}} = f(x-1) \). Shift the selected points horizontally \( c = 1 \) unit right. (Alternatively, add one to each \( x \) value in the table, and \( y \) values stay the same.)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( g_1(x) )</th>
</tr>
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<tbody>
<tr>
<td>-7</td>
<td>-2</td>
</tr>
<tr>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
</tr>
</tbody>
</table>

**Step 3:** Apply the reflection/magnification now. Graph: \( g_2(x) = -3 \cdot 3^{\sqrt[3]{x-1}} = -3 \cdot g_1(x) = -3 \cdot f(x-1) \). Multiply each \( y \)-coordinate of each point from the graph of \( g_1 = f(x-1) \) by \( a = -3 \).
Use translations to graph: \( g(x) = -3 \cdot 3^{\sqrt[3]{x - 1}} + 1 \)

**Step 3:** Apply the reflection/magnification now. Graph:
\[ g_2(x) = -3 \cdot 3^{\sqrt[3]{x - 1}} = -3 \cdot g_1(x) = -3 \cdot f(x - 1). \]
Multiply each y-coordinate of each point from the graph of \( g_1 = f(x - 1) \) by \( a = -3 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( g_2(x) )</th>
</tr>
</thead>
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</tr>
<tr>
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<tr>
<td>2</td>
<td>-3</td>
</tr>
<tr>
<td>9</td>
<td>-6</td>
</tr>
</tbody>
</table>

**Step 4:** graph: \( g(x) = -3 \cdot 3^{\sqrt[3]{x - 1}} + 1 = -3 \cdot f(x - 1) + 1 \). Now shift each point up (vertically) one unit; ie. add one to each y value.
Step 4: graph: \( g(x) = -3 \cdot \sqrt[3]{x - 1} + 1 = -3 \cdot f(x - 1) + 1 \). Now shift each point up (vertically) one unit; ie. add one to each \( y \) value.

\[
\begin{array}{c|c}
  x & g(x) \\
  \hline
  -7 & 7 \\
  0 & 4 \\
  1 & 1 \\
  2 & -2 \\
  9 & -5 \\
\end{array}
\]

\( g(x) = -3 \cdot \sqrt[3]{x - 1} + 1 = -3 \cdot f(x - 1) + 1 \).
Analyze: \( g(x) = -3 \cdot 3 \sqrt[3]{x - 1} + 1 \)

\( g(x) = -3 \cdot 3 \sqrt[3]{x - 1} + 1 = -3 \cdot f(x - 1) + 1 \) has the following characteristics:

- **Domain:** \( x \in (-\infty, \infty) \)
- **Range:** \( y \in (-\infty, \infty) \)
- **\( g(x) \) is a decreasing function.**
  \( g(x) \downarrow \text{ for } x \in (-\infty, \infty) \)
- \( \lim_{x \to \infty} g(x) = -\infty \)
- \( \lim_{x \to -\infty} g(x) = \infty \)