Test 4 Name: Key

No Calculators or Computing Devices allowed! Use Algebraic Notation AND Show All of Your Work.

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1. (6 points) Find the inverse of $C = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix}$ if it exists.

1. ________

$$[C \,\vdots\, I_3] = \left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 2 & -3 & -6 & 0 & 1 & 0 \\ -3 & 6 & 15 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -2R_1 + R_2 \to R_2 \\ 3R_1 + R_3 \to R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 3 & 3 & 0 & 1 \end{array}\right] \begin{array}{l} \\ 2R_2 + R_1 \to R_1 \\ \frac{1}{3}R_3 \to R_3 \end{array} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1/3 \end{array}\right],$$

Now $-2R_3 + R_2 \to R_2$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 0 & -4 & 1 & -2/3 \\ 0 & 0 & 1 & 1 & 0 & 1/3 \end{array}\right] = [I_3 \,\vdots\, A^{-1}]$$

So, $A^{-1} = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & 2/3 \\ 1 & 0 & 1/3 \end{bmatrix}$

2. (a) (2 points) Write a matrix equation equivalent to the following system.

$$\begin{cases} 3x + 2y = 14 \\ x - 2y = 2 \end{cases}$$

$$\begin{bmatrix} 3 & 2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 14 \\ 2 \end{bmatrix}$$

(a) ____________

(b) (4 points) Find the inverse of the coefficient matrix, and use it to solve the system.

$(x, y) = (-4, 1)$

(b) ____________

$$A^{-1} = \frac{1}{-6-2}\begin{bmatrix} -2 & -2 \\ -1 & 3 \end{bmatrix} = -\frac{1}{8}\begin{bmatrix} -2 & -2 \\ -1 & 3 \end{bmatrix}$$

and

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{8}\begin{bmatrix} -2 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 14 \\ 2 \end{bmatrix} = -\frac{1}{8}\begin{bmatrix} -2(14) - 2(2) \\ -1(14) + 3(2) \end{bmatrix}$$

$$= -\frac{1}{8}\begin{bmatrix} -32 \\ -8 \end{bmatrix} = \begin{bmatrix} -32/_{-8} \\ -8/_{-8} \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

3. (5 points) Solve $\begin{Bmatrix} 2x + y = 1 \\ 3x + 4y = 14 \end{Bmatrix}$ using Cramer's Rule.

3. ____________

$X = \dfrac{D_x}{D}$ and $Y = \dfrac{D_y}{D}$ where $D = \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 8 - 3 = 5$

And $D_x = \begin{vmatrix} 1 & 1 \\ 14 & 4 \end{vmatrix} = 4 - 14 = -10$ and

$D_y = \begin{vmatrix} 2 & 1 \\ 3 & 14 \end{vmatrix} = 28 - 3 = 25$. Thus, $X = \dfrac{-10}{5} = -2$

and $y = \dfrac{25}{5}$

4. Let $A = \begin{bmatrix} 1 & -5 \\ -3 & 7 \end{bmatrix}$, $B = \begin{bmatrix} -2 & -6 \\ 2 & 7 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 3 & 1 \\ -2 & 7 & 2 \\ 0 & 2 & 4 \end{bmatrix}$

Carry out the indicated operation, or explain, using complete sentences, why it cannot be performed.

(a) (2 points) $A + B$ A and B don't have the same dimension so addn is not defined for A+B.

(b) (2 points) AB

A B

2×2 3×2

Can't be done the number of columns in A don't equal the number of rows in B.

(c) (2 points) $BA - 3A$

$$BA-3A = \begin{bmatrix} -2 & -6 \\ 2 & 7 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -5 \\ -3 & 7 \end{bmatrix} - 3\begin{bmatrix} 1 & -5 \\ -3 & 7 \end{bmatrix} = \begin{bmatrix} -2+18 & 10-42 \\ 2-21 & -10+49 \\ 1+0 & -5+0 \end{bmatrix} - 3\begin{bmatrix} 1 & -5 \\ -3 & 7 \end{bmatrix}$$

3×2 2×2

Subtraction is not defined; can't be done!

(d) (2 points) B^{-1} Inverses are only for square matrices. B doesn't have an inverse

(e) (2 points) $\det(B)$ Det(B) doesn't exist.

Determinants are for square matrices

5. (6 points) Find the partial fraction decomposition of $\dfrac{7x-2}{x^2-4}$.

5. ____________

$$\frac{7x-2}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$$

$$7x-2 = A(x+2) + B(x-2)$$

$$7x-2 = Ax + 2A + Bx - 2B$$

$$7x-2 = (A+B)x + (2A-2B)$$

$$\Rightarrow \begin{Bmatrix} 7 = A+B \\ -2 = 2A-2B \end{Bmatrix} = \begin{Bmatrix} A+B = 7 \\ A-B = -1 \end{Bmatrix}$$

$$= \begin{Bmatrix} A+B = 7 \\ 2A = 6 \end{Bmatrix} = \begin{Bmatrix} B = 4 \\ A = 3 \end{Bmatrix}$$

Check $\dfrac{3}{x-2} + \dfrac{4}{x+2} = \dfrac{3(x+2)+4(x-2)}{x^2-4} = \dfrac{3x+6+4x-8}{x^2-4}$ ✓

So $$\boxed{\frac{7x-2}{x^2-4} = \frac{3}{x-2} + \frac{4}{x+2}}$$

6. Only one of the following two matrices has an inverse.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 0 & 8 \\ 0 & 2 & 2 \end{bmatrix}$$

(a) (5 points) Find the determinant of each matrix. (a) $\boxed{\det(A) = -44 \quad \det(B) = 0}$

$$\det(A) = -0 + 2\begin{vmatrix} 2 & -1 \\ -2 & 6 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 \\ -2 & 5 \end{vmatrix}$$

$$= 2(12-2)\ 4(10+6) = 2(10) - 4(16)$$

$$= 20 - 64 = -44$$

$$\det(B) = -2\begin{vmatrix} 3 & 7 \\ 2 & 2 \end{vmatrix} + 0 - 8\begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix}$$

$$= -2(6-14) - 8(2)$$

$$= -2(-8) - 16$$

$$= 16 - 16 = 0$$

(b) (1 point) Use the determinants from part (a) to identify which matrix has an inverse.

A has an Inverse

(b) ________

B doesn't have an inverse.

7. Let $A = \begin{bmatrix} 2 & -5 \\ -6 & 2 \\ 2 & -8 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 \\ 3 & -4 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 0 & 1 \\ -2 & 4 & 6 \\ 2 & 2 & 5 \end{bmatrix}$

Carry out the indicated operation, or explain, using complete sentences, why it cannot be performed.

$3 \times 3 \quad 3 \times 2$

(a) (4 points) CA

$$= \begin{bmatrix} 3 & 0 & 1 \\ -2 & 4 & 6 \\ 2 & 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ -6 & 2 \\ 2 & -8 \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \\ d_{31} & d_{32} \end{bmatrix}$$

$d_{11} = 6 + 0 + 2 = 8$

$d_{12} = -15 + 0 - 8 = -23$

$d_{21} = -4 - 24 + 12 = -16$

$d_{22} = 10 + 8 - 48 = -30$

$d_{31} = 4 - 12 + 10 = 2$

$d_{32} = -10 + 4 + -40 = -46$

$$= \begin{bmatrix} 8 & -23 \\ -16 & -30 \\ 2 & -46 \end{bmatrix}$$

(b) (4 points) $2B - 3A$

$$= 2\begin{bmatrix} -1 & 3 \\ 3 & -4 \\ 1 & 0 \end{bmatrix} - 3\begin{bmatrix} 2 & -5 \\ -6 & 2 \\ 2 & -8 \end{bmatrix} = \begin{bmatrix} -2 & 6 \\ 6 & -8 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} -6 & 15 \\ 18 & -6 \\ -6 & 24 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 21 \\ 24 & -14 \\ -4 & 24 \end{bmatrix}$$

8. (6 points) Find the complete solution of the system, or show that no solution exists.

$$\begin{cases} x - y + 5z = -2 \\ 2x + y + 4z = 2 \\ 2x + 4y - 2z = 8 \end{cases}$$

8. ________

$$= \left[\begin{array}{ccc|c} 1 & -1 & 5 & -2 \\ 2 & 1 & 4 & 2 \\ 2 & 4 & -2 & 8 \end{array}\right] \begin{array}{l} -2R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 5 & -2 \\ 0 & 3 & -6 & 6 \\ 0 & 6 & -12 & 12 \end{array}\right] \begin{array}{l} \frac{1}{3}R_2 \to R_2 \\ \frac{1}{6}R_3 \to R_3 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 5 & -2 \\ 0 & 1 & -2 & 2 \\ 0 & 1 & -2 & 2 \end{array}\right] -R_2 + R_3 \to R_3 \quad = \left[\begin{array}{ccc|c} 1 & -1 & 5 & -2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

zero row $\Rightarrow$ inf. number of solns,

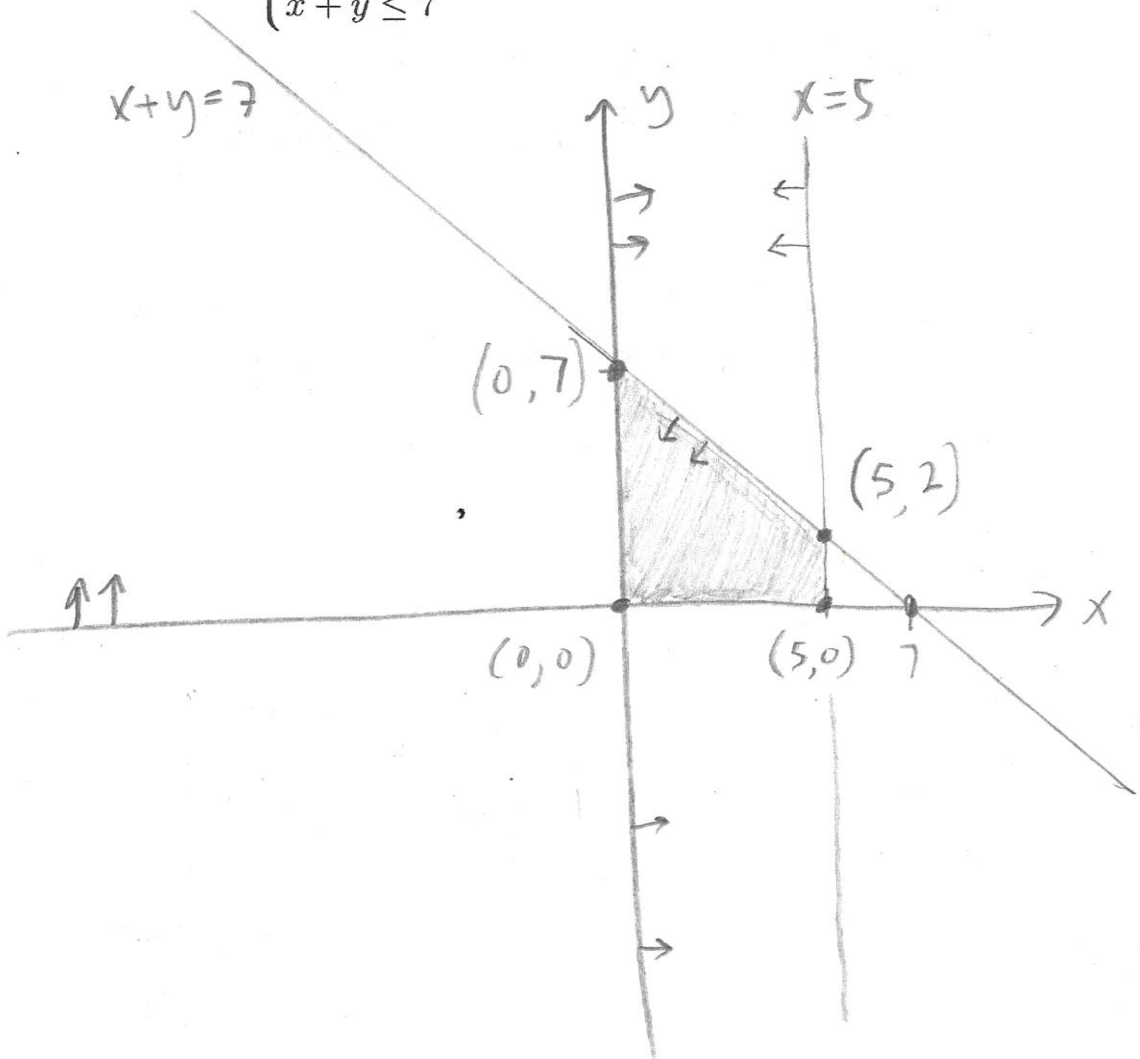
$$= \left\{ \begin{array}{l} x - y + 5z = -2 \\ y - 2z = 2 \end{array} \right\} = \left\{ \begin{array}{l} x = -2 + y - 5z \\ y = 2 + 2z \end{array} \right\}$$

$$= \left\{ \begin{array}{l} x = -2 + (2 + 2z) - 5z \\ y = 2 + 2z \end{array} \right\} = \left\{ \begin{array}{l} x = -3z \\ y = 2 + 2z \end{array} \right\}$$

$\boxed{\text{Soln set } \{ (x, y, z) = (-3z, 2+2z, z) \mid z \text{ is any real number} \}}$

9. (6 points) Sketch the graph (and label the vertices, or boundary intersections) of the solution set of ordered pairs of the system.

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ x \leq 5 \\ x + y \leq 7 \end{cases}$$



10. (6 points) Use Gaussian elimination to find the complete solution of the system, or show that no solution exists.

Replace eqn 2 with $-2\cdot$eqn 1 + eqn 2

$$\begin{cases} x - y + 2z = 0 \\ 2x - 4y + 5z = -5 \\ 2y - 3z = 5 \end{cases}$$

$(x,y,z) = (2.5, 2.5, 0)$

10. ________

$$= \begin{Bmatrix} x - y + 2z = 0 \\ -2y + z = -5 \\ 2y - 3z = 5 \end{Bmatrix}$$

Replace eqn 3 with eqn 2 + eqn 3.

$$= \begin{Bmatrix} x - y + 2z = 0 \\ -2y + z = -5 \\ -2z = 0 \end{Bmatrix}$$

Replace eqn 2 with $-\frac{1}{2}\cdot$ eqn 2. Replace eqn 3 with $-\frac{1}{2}\cdot$ eqn 3.

$$= \begin{Bmatrix} x - y + 2z = 0 \\ y - \frac{1}{2}z = \frac{5}{2} \\ z = 0 \end{Bmatrix}$$

System is in Row-echelon form. Now back-substitute

$$= \begin{Bmatrix} x = 5/2 \\ y = \frac{5}{2} \\ z = 0 \end{Bmatrix} = \begin{Bmatrix} x = 2.5 \\ y = 2.5 \\ z = 0 \end{Bmatrix}$$

Check

(1) $\frac{5}{2} - \frac{5}{2} + 2\cdot 0 = 0$ ✓

(2) $2\cdot\frac{5}{2} - \cancel{4}\,\frac{5}{\cancel{2}} + 5\cdot 0 = -5$

$5 - 10 + 0 = -5$ ✓

(3) $2\cdot\frac{5}{2} - 3\cdot 0 = 5$

$5 - 0 = 5$ ✓

11. (6 points) Write the given system as an augmented matrix. Use Elementary Row Operations to derive equivalent matrices and find the complete solution of the system, or show that no solution exists.

$$\begin{cases} x - 3y + 2z = 12 \\ 2x - 5y + 5z = 14 \\ x - 2y + 3z = 20 \end{cases}$$

11. ________

$$= \left[\begin{array}{ccc|c} 1 & -3 & 2 & 12 \\ 2 & -5 & 5 & 14 \\ 1 & -2 & 3 & 20 \end{array}\right] \begin{array}{l} \\ -2R_1 + R_2 \to R_2 \\ -R_1 + R_3 \to R_3 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & -3 & 2 & 12 \\ 0 & 1 & 1 & -10 \\ 0 & 1 & 1 & 8 \end{array}\right] \begin{array}{l} \\ \\ -R_2 + R_3 \to R_3 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & -3 & 2 & 12 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 0 & 18 \end{array}\right]$$

System is in row-echelon form.
Last row says

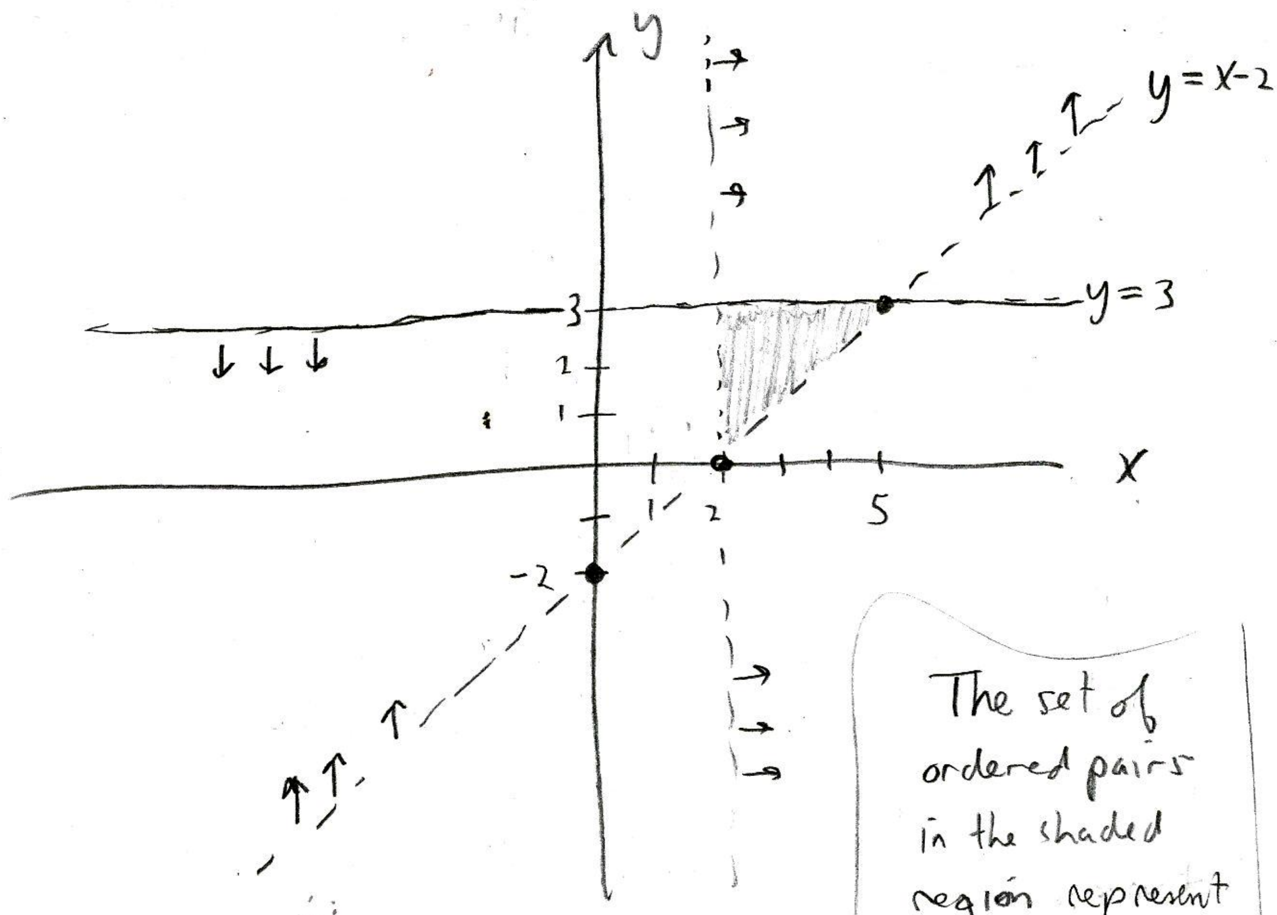
$0x + 0y + 0z = 18$, or

$0 = 18$, a false statement.

Since a false statement was derived, the system is inconsistent. There is no solution to the system of equations.

12. (6 points) Sketch the graph (and label the vertices, or boundary intersections) of the solution set of ordered pairs of the system.

$$\begin{cases} x - y < 2 \\ x > 2 \\ y \leq 3 \end{cases} = \begin{cases} y > x - 2 \\ x > 2 \\ y \leq 3 \end{cases}$$



The set of ordered pairs in the shaded region represent the solution set.

13. (6 points) Find the partial fraction decomposition of $\frac{3x-4}{x^3+4x^2}$.

13. ________

$$\frac{3x-4}{x^2(x+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+4}$$

$$x^2\cdot(x+4)\left(\frac{3x-4}{x^2(x+4)}\right) = x^2(x+4)\cdot\left(\frac{A}{x}\right) + x^2(x+4)\cdot\left(\frac{B}{x^2}\right) + x^2(x+4)\cdot\left(\frac{C}{x+4}\right)$$

$$3x-4 = x(x+4)A + (x+4)B + x^2\cdot C$$

$$\boxed{3x-4 = Ax(x+4) + B(x+4) + Cx^2 \quad \text{Basic Equation}}$$

When $x=0$ this is $-4 = 4B$, or $\boxed{B=-1}$. When $x=-4$, the basic equation is $-16 = 16C$, or $\boxed{C=-1}$. To get A we replace x in the basic eqn with any number other than -4 or 0. Let $x=1$ in the basic eqn then $-1 = 5A-5-1$, or $5A=5$, or $A=1$. So, $\boxed{\frac{3x-4}{x^3+4x^2} = \frac{1}{x} - \frac{1}{x^2} - \frac{1}{x+4}}$

Check $\frac{1}{x}\cdot\frac{x(x+4)}{x(x+4)} - \frac{1}{x^2}\cdot\frac{x+4}{x+4} - \frac{1}{x+4}\cdot\frac{x^2}{x^2}$

$$= \frac{x^2+4x-x-4-x^2}{x^2(x+4)} = \frac{3x-4}{x^3+4x} \checkmark$$

14. (6 points) Find the partial fraction decomposition of $\frac{2x-3}{x^3+3x}$.

$$\frac{2x-3}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$$

14. $\boxed{\frac{2x-3}{x^3+3x} = \frac{-1}{x} + \frac{x+2}{x^2+3}}$

$$x\cdot(x^2+3)\,\frac{2x-3}{x(x^2+3)} = x(x^2+3)\cdot\frac{A}{x} + x(x^2+3)\cdot\frac{Bx+C}{x^2+3}$$

$$\boxed{2x-3 = A(x^2+3) + x(Bx+C) \quad \text{Basic Eqn}}$$

Let $x=0$, then $-3=3A$, or $\boxed{A=-1}$ then the basic eqn

reduces to $2x-3 = -1(x^2+3) + x(Bx+C)$, or

$x^2+2x = Bx^2 + Cx$. Let $x=1$ and $x=-1$, this

gives us two eqns in B and C.

$$\begin{Bmatrix} 3 = B+C \\ -1 = B-C \end{Bmatrix} = \begin{Bmatrix} B+C=3 \\ B-C=-1 \end{Bmatrix}$$ Replace eqn 2 with eqn 1 + eqn 2

$$= \begin{Bmatrix} B+C=3 \\ 2B=2 \end{Bmatrix} = \begin{Bmatrix} B+C=3 \\ B=1 \end{Bmatrix} = \begin{Bmatrix} 1+C=3 \\ B=1 \end{Bmatrix}$$

$$= \begin{Bmatrix} B=1 \\ C=2 \end{Bmatrix}$$

Check $\frac{-1}{x}\cdot\frac{x^2+3}{x^2+3} + \frac{x+2}{x^2+3}\cdot\frac{x}{x}$

$$\frac{-x^2-3+x^2+2x}{x(x^2+3)} = \frac{2x-3}{x^3+3x} \checkmark$$