Polynomial Division

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Dividing Polynomials

Simple Case: Division by a Monomial

Example: Divide $\dfrac{6x^3 - 9x^2 + 12x}{3x}$

Solution

$$\frac{6x^3 - 9x^2 + 12x}{3x} = \frac{1}{3x} \cdot (6x^3 - 9x^2 + 12x) = \frac{1}{3x} \cdot 6x^3 + \frac{1}{3x} \cdot (-9x^2) + \frac{1}{3x} \cdot 12x$$

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$$\frac{6x^3 - 9x^2 + 12x}{3x} = \frac{1}{3x} \cdot (6x^3 - 9x^2 + 12x) = \frac{1}{3x} \cdot 6x^3 + \frac{1}{3x} \cdot (-9x^2) + \frac{1}{3x} \cdot 12x$$

$$= \frac{6x^3}{3x} - \frac{9x^2}{3x} + \frac{12x}{3x} = 2x^2 - 3x + 4$$

Try This One! Divide $\dfrac{27x^4y^7 - 81x^5y^3}{-9x^3y^2}$ to lowest terms.

Dividing Polynomials

Procedure

Whenever the denominator is not a monomial, or a factor of the numerator, or if the numerator is not factorable, the previous method won't work. So, instead we use long division of polynomials, a method similar to long division of whole numbers.

Theorem (Division Algorithm:)

Suppose D and P are polynomial expressions of variable x, with $D \neq 0$, and suppose that D is less than the degree of P. Then there exist unique polynomials Q and R, where R is either 0 or has degree less than the degree of D, such that

$$P = Q \cdot D + R \ \text{ or, equivalently } \ \frac{P}{D} = Q + \frac{R}{D}$$

In words, we have

dividend = (quotient)·(divisor) + remainder

or

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}.$$

Long Division of Polynomials

Dividing $\dfrac{x^3+2x^2-x-2}{x-1}$ is equivalent to the long division problem and solution:

$$
\begin{array}{rrl}
 & x^2 + 3x + 2 & \leftarrow \text{Quotient, Q(x)} \\
\text{Divisor, D(x)} \twoheadrightarrow \quad x-1 & \overline{)\,x^3 + 2x^2 - x - 2} & \leftarrow \text{Dividend, P(x)} \\
 & \underline{x^3 - x^2} & \\
 & 3x^2 - x - 2 & \\
 & \underline{3x^2 - 3x} & \\
 & 2x - 2 & \\
 & \underline{2x - 2} & \\
 & 0 & \leftarrow \text{Remainder, R(x)}
\end{array}
$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Set the problem up for long division. Write the dividend in descending ord
insert zero placeholders for any missing polynomial terms, if necessary.

$$x - 1 \,\overline{\big)\; x^3 \;+\; 2x^2 \;-\; x \;-\; 2}$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Divide the leading term of the dividend by the leading term in the divisor.

$$x - 1 \overline{\big)\, x^3 + 2x^2 - x - 2}$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Divide the leading term of the dividend by the leading term in the divisor.

That is, divide $\dfrac{x^3}{x}$

$$x - 1 \,\overline{\big)\; x^3 + 2x^2 - x - 2}$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Divide the leading term of the dividend by the leading term in the divisor. That is, divide $\dfrac{x^3}{x} = x^2$. Write this above the x^2 term of the dividend.

$$\begin{array}{r|l} & \quad\quad\quad\; x^2 \\ \hline x - 1 & x^3 + 2x^2 - x - 2 \end{array}$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Multiply $x^2 \cdot (x - 1)$ and list the result below $x^3 + 2x^2 - x - 2$

$$\begin{array}{r|l} & x^2 \\ \cline{2-2} x - 1 & x^3 + 2x^2 - x - 2 \end{array}$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Multiply $x^2 \cdot (x - 1) = x^3 - x^2$

$$
\begin{array}{r|l}
 & \quad\quad\quad\quad x^2 \\
\cline{2-2}
x - 1 & x^3 + 2x^2 - x - 2 \\
 & x^3 - x^2
\end{array}
$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Subtract. CHANGE THE SIGNS AND ADD

$$
\begin{array}{r|rrrrrrr}
 & & & x^2 & & & & \\
\hline
x - 1 & x^3 & + & 2x^2 & - & x & - & 2 \\
 & -x^3 & + & x^2 & & & & \\
\hline
 & 0 & + & 3x^2 & & & &
\end{array}
$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Bring $-x - 2$ down.

$$
\begin{array}{r|rrrrrr}
 & & & x^2 & & & \\ \cline{2-7}
x - 1 & x^3 & + & 2x^2 & - & x & - \; 2 \\
 & -x^3 & + & x^2 & & & \\ \cline{2-7}
 & & & 3x^2 & - & x & - \; 2
\end{array}
$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Repeat the process using $3x^2 - x - 2$ as the dividend. Add $\dfrac{3x^2}{x} = 3x$ to the quotient.

$$
\begin{array}{r|rrrr}
 & & x^2 & +\ 3x & \\
\cline{2-5}
x - 1 & x^3 + & 2x^2 - & x - & 2 \\
 & -x^3 + & x^2 & & \\
\cline{2-5}
 & & 3x^2 - & x - & 2
\end{array}
$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Multiply $3x \cdot (x - 1)$ and list the result below $3x^2 - x - 2$.

$$\begin{array}{r|rrrr} & & x^2 & +\, 3x & \\ \hline x - 1 & x^3 + & 2x^2 & - x & - 2 \\ & -x^3 + & x^2 & & \\ \hline & & 3x^2 & - x & - 2 \end{array}$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Multiply $3x \cdot (x - 1) = 3x^2 - 3x$

$$
\begin{array}{r|rrrrrrr}
 & & & x^2 & + & 3x & & \\
\hline
x - 1 & x^3 & + & 2x^2 & - & x & - & 2 \\
 & -x^3 & + & x^2 & & & & \\
\hline
 & & & 3x^2 & - & x & - & 2 \\
 & & & 3x^2 & - & 3x & &
\end{array}
$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Subtract. CHANGE THE SIGNS AND ADD.

$$
\begin{array}{r|rrrr}
 & & x^2 & +\ 3x & \\
\hline
x - 1 & x^3 \ + & 2x^2 & -\ x & -\ 2 \\
 & -x^3 \ + & x^2 & & \\
\cline{2-5}
 & & 3x^2 & -\ x & -\ 2 \\
 & & -3x^2 & +\ 3x & \\
\cline{3-5}
 & & 0 & +\ 2x & \\
\end{array}
$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Bring -2 down.

$$
\begin{array}{r|rrrr}
 & & x^2 & + 3x & \\
\cline{2-5}
x - 1 & x^3 & + 2x^2 & - x & - 2 \\
 & -x^3 & + x^2 & & \\
\cline{2-5}
 & & 3x^2 & - x & - 2 \\
 & & -3x^2 & + 3x & \\
\cline{3-5}
 & & & 2x & - 2
\end{array}
$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Repeat the process using $2x - 2$ as the dividend. Add $\dfrac{2x}{x} = 2$ to the quo

$$
\begin{array}{r|rrrr}
 & & x^2 & + \; 3x & + \; 2 \\ \hline
x - 1 & x^3 & + \; 2x^2 & - \; x & - \; 2 \\
 & -x^3 & + \; x^2 & & \\ \cline{2-5}
 & & 3x^2 & - \; x & - \; 2 \\
 & & -3x^2 & + \; 3x & \\ \cline{3-5}
 & & & 2x & - \; 2
\end{array}
$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Multiply $2 \cdot (x - 1)$ and list the result below $2x - 2$.

$$
\begin{array}{r|rrrr}
 & & x^2 & +\ 3x & +\ 2 \\
\hline
x - 1 & x^3 & +\ 2x^2 & -\ x & -\ 2 \\
 & -x^3 & +\ x^2 & & \\
\hline
 & & 3x^2 & -\ x & -\ 2 \\
 & & -3x^2 & +\ 3x & \\
\hline
 & & & 2x & -\ 2
\end{array}
$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Multiply $2 \cdot (x - 1) = 2x - 2$

$$
\begin{array}{r|rrrr}
 & & x^2 & + \; 3x & + \; 2 \\
\hline
x - 1 & x^3 & + \; 2x^2 & - \; x & - \; 2 \\
 & -x^3 & + \; x^2 & & \\
\hline
 & & 3x^2 & - \; x & - \; 2 \\
 & & -3x^2 & + \; 3x & \\
\hline
 & & & 2x & - \; 2 \\
 & & & 2x & - \; 2
\end{array}
$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Subtract. CHANGE THE SIGNS AND ADD

$$
\begin{array}{r|rrrr}
 & x^2 & + & 3x & + 2 \\
\hline
x - 1 & x^3 + & 2x^2 & - x & - 2 \\
 & -x^3 + & x^2 & & \\
\hline
 & & 3x^2 & - x & - 2 \\
 & & -3x^2 & + 3x & \\
\hline
 & & & 2x & - 2 \\
 & & & -2x & + 2 \\
\hline
 & & & & 0
\end{array}
$$

Hence the remainder, $R(x) = 0$, and

$$\frac{x^3 + 2x^2 - x - 2}{x - 1} = x^2 + 3x + 2$$

How does one know when the long division process is finished?

It is not always the case that the remainder will be zero when dividing two polynomials. So, how does one know when the long division process of polynomials is over? When the degree of the remainder is less than the degree of the divisor.

Recall that the degree of a polynomial expression is the largest power of x that occurs amongst the terms in the expression.

$$
\begin{array}{r|rrrr}
 & x^2 & +\ 3x & +\ 2 & \\
\cline{2-5}
x-1 & x^3 & +\ 2x^2 & -\ x & -\ 2 \\
 & x^3 & -\ x^2 & & \\
\cline{2-5}
 & & 3x^2 & -\ x & -\ 2 \\
 & & 3x^2 & -\ 3x & \\
\cline{3-5}
 & & & 2x & -\ 2 \\
 & & & 2x & -\ 2 \\
\cline{4-5}
 & & & & 0
\end{array}
$$