2.1 Functions **Name:** ______________________

1. A function f is defined by the formula $f(x) = x^2 + 4$.
   (a) Express in words how f acts on the input x to produce output $f(x)$.
   (b) Evaluate $f(3)$, $f(-2)$ and $f(\sqrt{5})$.
   (c) Find the domain and range of f

2. Suppose $f(x) = 3x^2 + x - 5$. Evaluate the function at the values indicated below.
   (a) $f(-2)$ (a) ____________
   (b) $f(0)$ (b) ____________
   (c) $f(4)$ (c) ____________
   (d) $f(\frac{1}{2})$ (d) ____________

3. Suppose $C(x) = \left\{ \begin{array}{ll} 39 & \text{if } 0 \le x \le 400 \\ 39 + 0.20(x - 400) & \text{if } x > 400 \end{array} \right\}$. Evaluate the piecewise defined function at the values indicated below.
   (a) $C(100)$ (a) ____________
   (b) $C(400)$ (b) ____________
   (c) $C(480)$ (c) ____________

4. Suppose $g(x) = \left\{ \begin{array}{ll} 2x & \text{if } x < 0 \\ x + 3 & \text{if } 0 \le x \le 2 \\ (x-1)^2 & \text{if } x > 2 \end{array} \right\}$. Evaluate the piecewise defined function at the values indicated below.
   (a) $g(-5)$ (a) ____________
   (b) $g(0)$ (b) ____________
   (c) $g(1)$ (c) ____________
   (d) $g(2)$ (d) ____________
   (e) $g(5)$ (e) ____________

5. If an astronaut weighs 130 pounds on the surface of the earth, then her weight when when she is h miles above the earth is given by $w(h) = 130\left(\frac{3960}{3960 + h}\right)^2$.

(a) What is her weight when she is 100 mi above earth? (a) ______________

(b) Use the graphing calculator to construct a table of values for the function w that gives her weight at heights from 0 to 500 mi, in increments of 100. What can you conclude from the table?

Definition 1. *If n is any positive integer, then the principal n^{th} root of a is defined as follows:*

$$\sqrt[n]{a} = b \quad \textit{means} \quad b^n = a$$

If n is even, we must have $a \geq 0$ and $b \geq 0$.

Definition 2. *The domain of a function is the set of all real-valued inputs which produce real-valued outputs .*

Find the domain of each function given below. Express the domain set using interval notation.

6. $C(x) = \dfrac{1}{x - 5}$

9. $g(x) = \sqrt[3]{x}$

7. $h(x) = \sqrt{x}$

10. $f(x) = \dfrac{2}{\sqrt[4]{x}}$

8. $f(x) = \dfrac{2}{\sqrt{x}}$

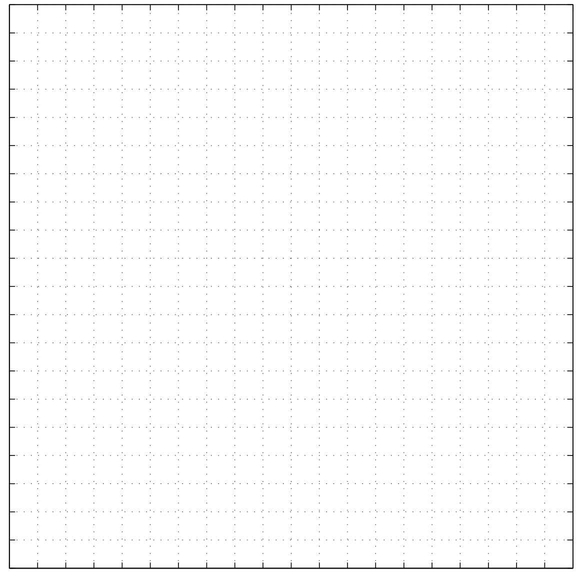
11. $h(x) = \sqrt{9 - x^2}$

2.2 Function Graphs

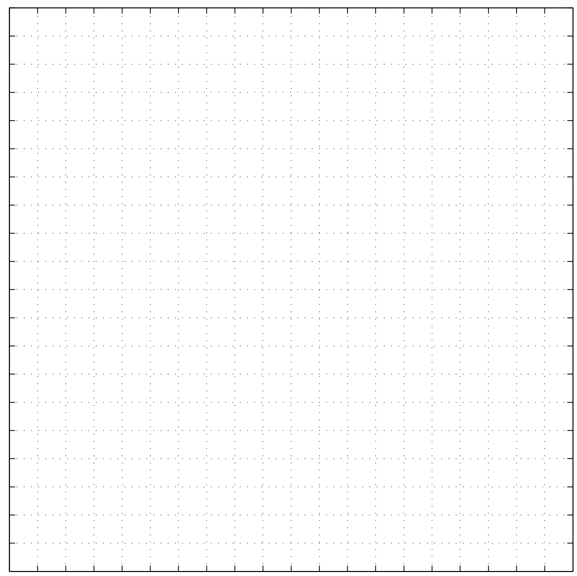
Name: ________________________

Directions: Fill in each table of values then graph each point in your table and the given function.

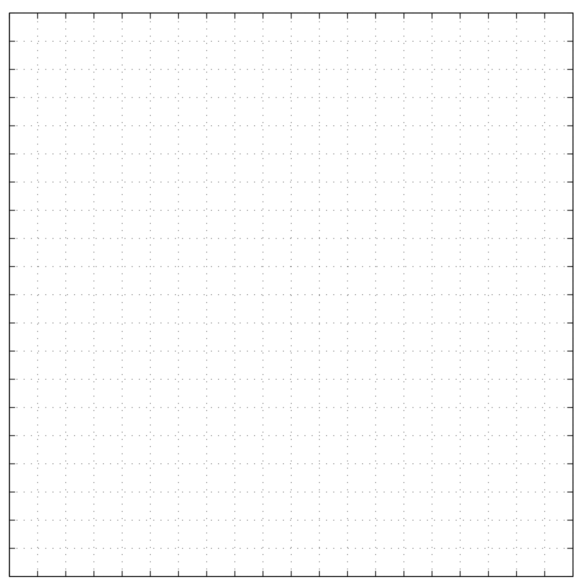
x	$f(x) = x^2$
0	
$\pm 1/2$	
± 1	
± 2	
± 3	



x	$f(x) = x^3$
0	
$1/2$	
1	
2	
$-1/2$	
-1	
-2	



x	$f(x) = \sqrt{x}$
0	
1	
2	
3	
4	
5	
6	
9	



Definition 3. *A* *piecewise defined function* *is defined by different formulas on different parts of its domain.*

1. Graph $C(x) = \left\{ \begin{array}{ll} 3 - x & \text{if } x \leq -2 \\ 1 - x^2 & \text{if } x > -2 \end{array} \right\}$

Graph $h(x) = \left\{ \begin{array}{ll} 2 & \text{if } x < -1 \\ \sqrt{1 - x^2} & \text{if } -1 \leq x \leq 1 \\ 1 + x & \text{if } x > 1 \end{array} \right\}$

3. Graph $f(x) = |x| = \left\{ \begin{array}{ll} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{array} \right\}$

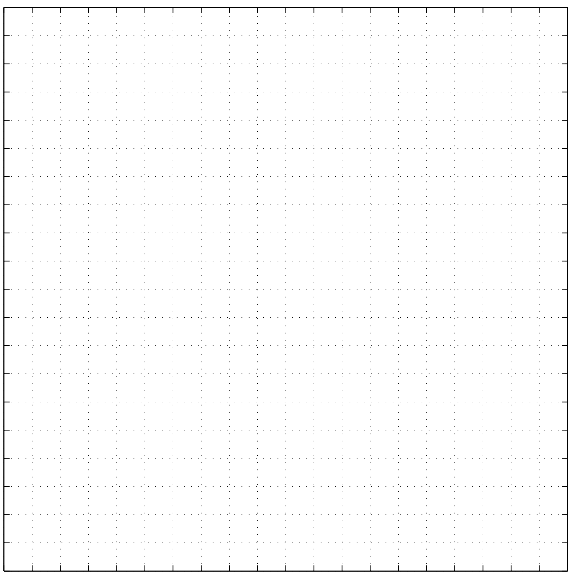
4. Evaluate and simplify the expression $\dfrac{f(a+h)-f(a)}{h}$ for $f(x) = 2x^2 - 1$. Assume a, h and $a+h$ are real numbers in the domain of f and that $h \neq 0$. Hint: You know you are finished when you can replace h with zero and not get a division by zero.

5. Evaluate and simplify the expression $\dfrac{f(a+h)-f(a)}{h}$ for $f(x) = \dfrac{3x}{1-x}$.

6. Use your graphing calculator to plot $f(x) = x^3 - 8x^2$. Use a viewing rectangle with size $[-4, 10]$ by $[-100, 100]$.

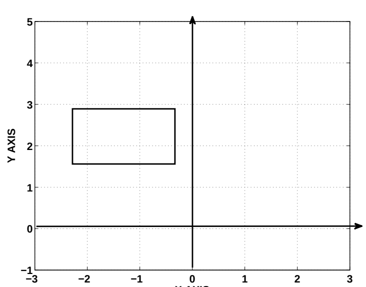
7. Use your graphing calculator to plot the piecewise function defined in question 2. (Hint pg. 155)

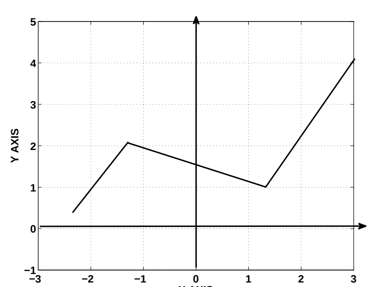
8. The cost of a long-distance daytime phone call from Toronto, Canada to Mumbai, India is 69 cents for the first minute and 58 cents for each additional minute (or part of a minute). Sketch the graph of the cost C (in dollars) of a phone call as a function of time t (in minutes).

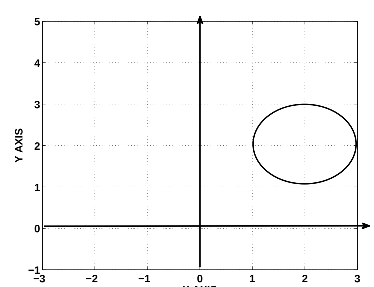


9. Does the equation $x^2 + y^2 = 16$ define y as a function of x?

10. Use the Vertical Line Test to determine if the given graphs are the graphs of a functions.







2.5 Transformations

Name: ______________________

Directions: Sketch the graph of the function, not by plotting points, but by starting with the graph of a standard function and applying transformations. Label at least 3 points on your final graph.

1. Graph $C(x) = 2|x + 2| - 3$

2. Graph $h(x) = \sqrt{x + 4} - 3$

Determine whether the function f is even, odd, or neither.

3. $f(x) = 2 - x^3 - x^7$

4. $f(x) = 2x^2 - 1$

Determine whether the function f is even, odd, or neither.

5. $f(x) = 1 - \sqrt[5]{x}$

A function f is given, and the indicated transformations are applied to its graph (in the given order). Write the equation for the final transformed graph.

6. $f(x) = \sqrt{9 - x^2}$; stretch vertically by a factor of 5, shift downward 10 units and shift 4 units right.

Directions: Sketch by using transformations. Label at least 3 points on your final graph.

7. Graph $R(x) = -2(x + 3)^2 - 3$

8. Graph $P(x) = 2 - \sqrt{1 - x^2}$

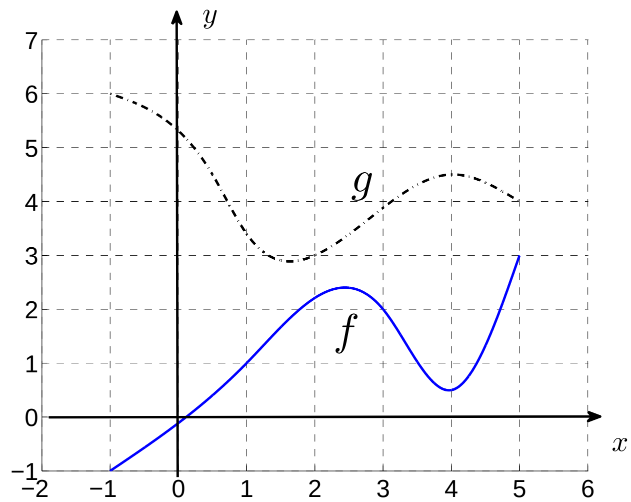
2.6 Combining Functions **Name:** ____________________

1. **Use the graph to find the indicated functional values.**
   (a) $(f+g)(3)$

   (b) $(f-g)(-1)$

   (c) $\dfrac{f}{g}(5)$

   (d) Find the domain and range of f



Find $f+g$, $f-g$, fg, and f/g and their domains.

2. $f(x) = x^2 - 14x$ and $g(x) = x + 14$.

3. $f(x) = \sqrt{7-x}$ and $g(x) = \sqrt{7+x}$

Find $f+g$, $f-g$, fg, and f/g and their domains.

4. $f(x) = \dfrac{7}{x-14}$ and $g(x) = \dfrac{15}{x+8}$

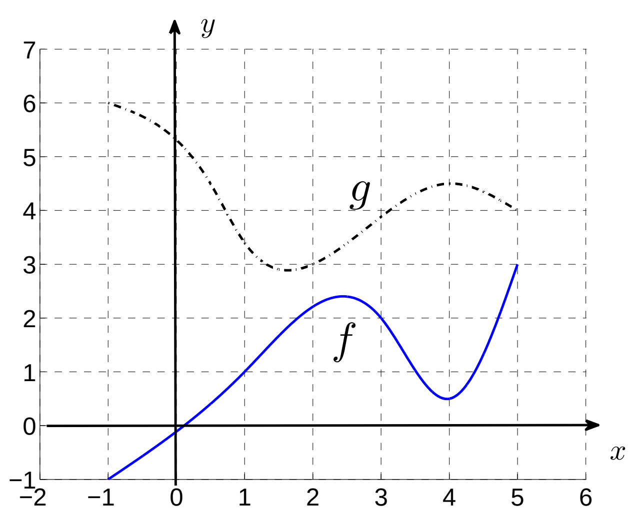
5. **Use the graph to find the indicated functional values.**

   (a) $f(g(3))$

   (b) $f(g(-1))$

   (c) $g(f(3))$

   (d) $g(g(3))$



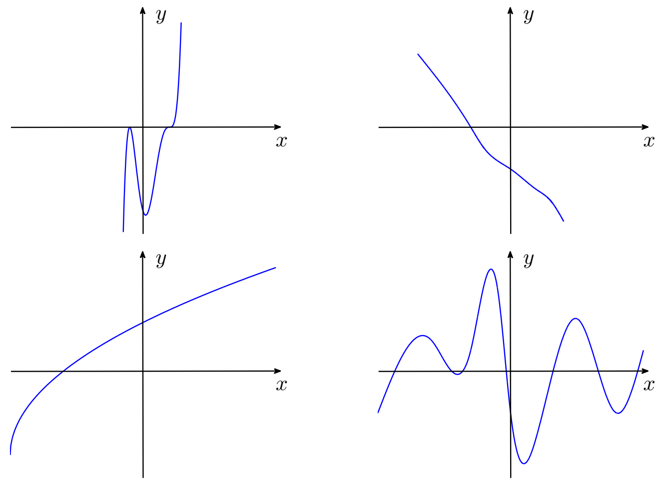
Find $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$ and their domains.

6. $f(x) = x^2 + 7$ and $g(x) = x - 7$.

7. $f(x) = \dfrac{1}{x}$ and $g(x) = 4x - 9$.

2.7 Inverse Functions **Name:** ________________________

1. Determine whether each function f is one-to-one.



Determine whether f is one-to-one.

2. $f(x) = 2 - x - x^2$

4. $f(x) = 2(x-3)^4$

3. $f(x) = 4 + (x-3)^5$

5. $f(x) = x^{2/3}$

Assume that f is one-to-one

7. If $f(3) = -56$, find $f^{-1}(-56)$ and $(f(3))^{-1}$

8. If $f^{-1}(2) = 21$, find $f(21)$ and $(f^{-1}(2))^{-1}$

Find the inverse function of f.

10. $f(x) = \dfrac{3x}{x-2}$

11. $f(x) = (2 - x^3)^5$

12. $f(x) = x^3 + 5$

13. $f(x) = x^2 + x \quad \text{for } x \geq \dfrac{1}{2}$

3.1 Quadratic Functions **Name:** ____________________

1. Find the vertex of $g(x) = 3(x-5)^2 + 7$. Does f open up or down?

1. ____________

2. What is the range of g?

2. ____________

3. Find the vertex of $g(x) = -2(x+8)^2 - 4$. Does f open up or down?

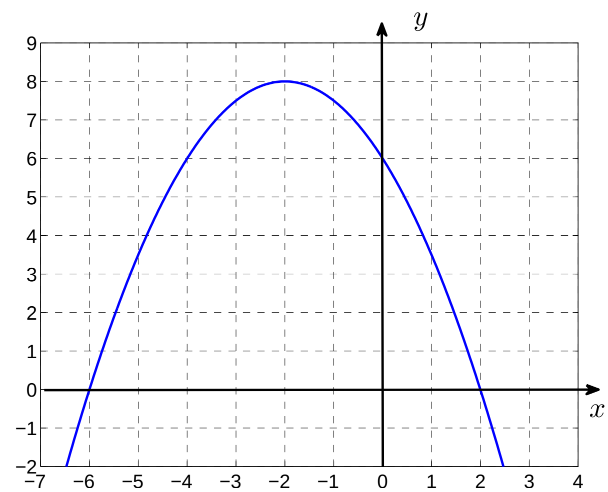
3. ____________

4. What is the range of g?

4. ____________

The graph of a quadratic function f is given. Find the coordinates of the vertex. Find the maximum or minimum of f. Find the intervals on which the function is increasing and on which the function is decreasing. Find the domain and range of f.

5. $f(x) = -\frac{1}{2}x^2 - 2x + 6$



Express the quadratic function in standard (vertex) form.

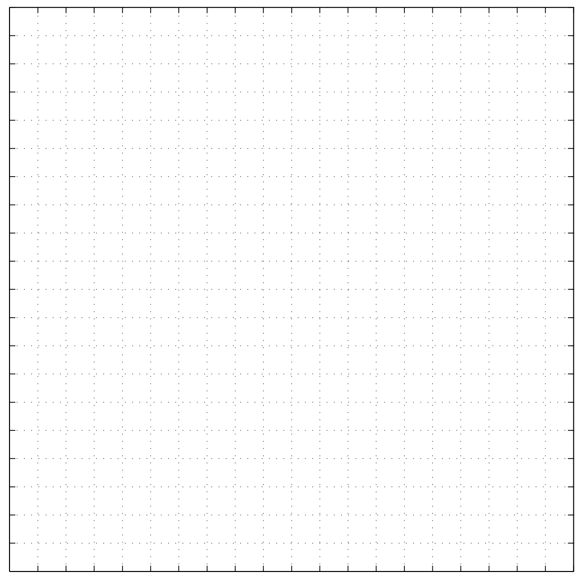
6. $g(x) = x^2 + 8x - 23$

7. $g(x) = 2x^2 + x - 6$

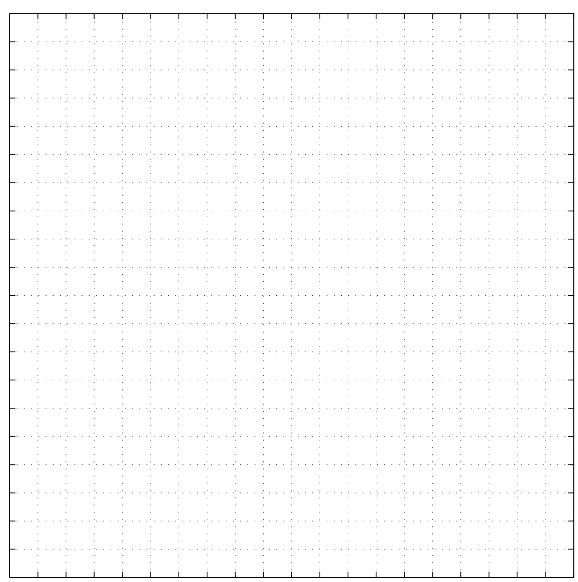
8. Find a function whose graph is a parabola with vertex $(-3, 5)$ and that passes through $(3, 2)$.

A quadratic function is given. Express the quadratic function in standard form. Find its vertex and its $x-$ and y- intercept(x). Sketch its graph.

9. $C(x) = -x^2 - 4x + 4$



10. $h(x) = 2x^2 + 7x - 5$



3.3 Dividing Polynomials **Name:** ________________________

1. Divide $\dfrac{6x^3 - 9x^2 + 12x}{3x}$

1. ____________

2. Divide $\dfrac{27x^4y^7 - 81x^5y^3}{-9x^3y^2}$

2. ____________

3. Simplify $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$. Hint: Factor the numerator first (using the grouping technique) then divide out any common factors.

3. ____________

4. Use long division to divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$.

4. ____________

Find the quotient and remainder using long division.

5. $\dfrac{4x^2 - 3x - 7}{2x - 1}$

6. $\dfrac{9x^2 - x + 5}{3x^2 - 7x}$

7. $\dfrac{2x^4 - x^3 + 9x^2}{x^2 + 4}$

8. $\dfrac{x^3 + x^2 - 10x + 8}{x - 3}$

Find the quotient and remainder using synthetic division.

9. $\dfrac{x^3 + x^2 - 10x + 8}{x - 3}$

11. $\dfrac{x^3 - 7x + 3}{x + 4}$

10. $\dfrac{9x^2 - x + 5}{x + 5}$

12. $\dfrac{2x^3 + 5x^2 - 10x}{x - 3}$

Use synthetic division and the Remainder Theorem to evaluate $f(c)$.

13. $f(x) = x^3 - 8x^2 + 9x - 2;$
$c = -1$

14. $f(x) = 2x^5 - 13x^3 + 8x^2;$
$c = -4$

Use the Factor Theorem to show that $x - c$ is a factor of $f(x)$ for the given value of c.

15. $f(x) = x^3 + 2x^2 - 3x - 10;$
$c = 2$

16. $f(x) = x^4 + 3x^3 - 16x^2 - 27x + 63;$
$c = -3$

17. Use the Factor Theorem to find a polynomial function of degree 4 with zeros at $x = -2, 0, 2, 4$.

17. ____________

3.4 Rational Zeros Theorem **Name:** ________________________

Find all rational zeros of the polynomial, and write the polynomial in factored form.

1. $f(x) = x^3 - x^2 + 14x - 8$

2. $f(x) = 2x^4 - x^3 - 19x^2 + 9x + 9$

3. $f(x) = 6x^3 + 11x^2 - 3x - 2$

4. $f(x) = 6x^4 - 7x^3 - 12x^2 + 3x + 2$

Use Descartes' Rule of Signs to determine how many positive and how many negative real zeros the polynomial can have. Then determine the possible total number of real zeros.

5. $f(x) = x^3 - x^2 + 14x - 8$

6. $f(x) = x^4 - x^2 + 14x - 8$

3.2 Graphing Polynomials **Name: __________________________**

Use the Leading Coefficient Test to determine the end behavior of each polynomial.

1. $f(x) = x^3 - x^2 + 14x - 8$

2. $f(x) = 2x^4 - x^3 - 19x^2 + 9x + 9$

3. $f(x) = -6x^3 + 11x^2 - 3x - 2$

4. $f(x) = -6x^4 - 7x^3 - 12x^2 + 3x + 2$

Solve the following polynomial inequalities.

5. $f(x) < 0$ for $f(x) = (x - 2)^2(x - 5)$

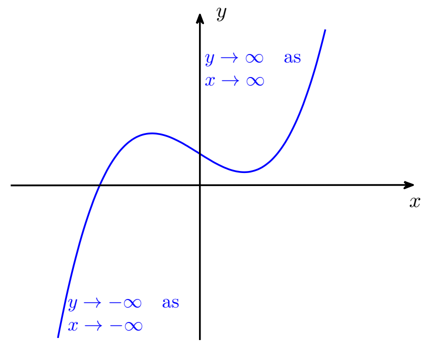
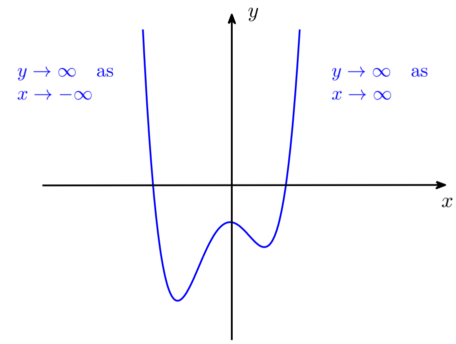
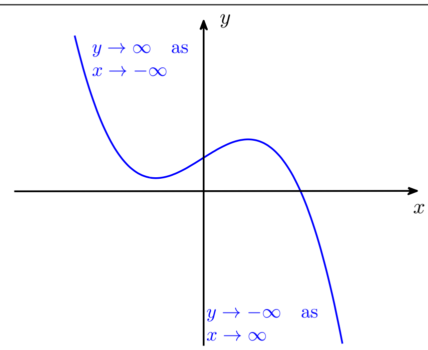
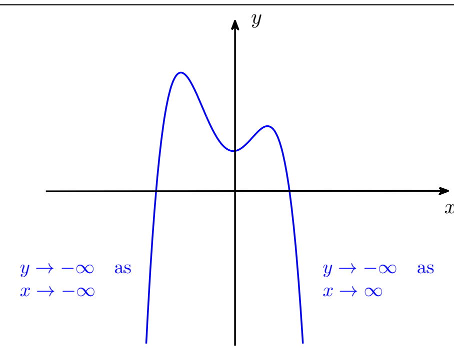
6. $f(x) > 0$ for $f(x) = x(x+1)^2(x-1)^3$

7. $-x^3 + 2x^2 + 4x \geq 8$

8. $x^5 - x^4 + 3x^3 - 3x^2 - 4x + 4 > 0$

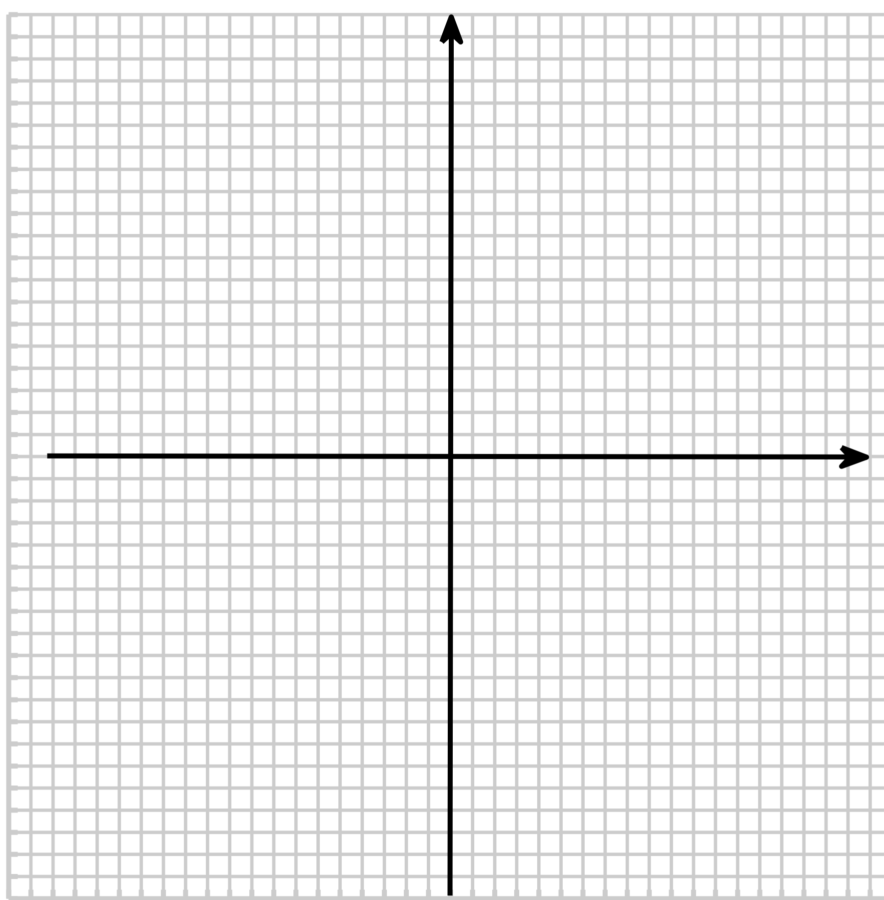
GRAPHING POLYNOMIAL FUNCTIONS

1. Determine if the graph has any symmetry. Locate the y intercept.
2. Factor the polynomial and find the zeros.
3. Determine the x intervals for which $f(x) > 0$ (is above the y axis) and $f(x) < 0$ (is below the y axis).
4. Plot the zeros on the real number line. Label each zero as being either odd or even.
5. Make a table of values. Mark the end behavior of the graph.

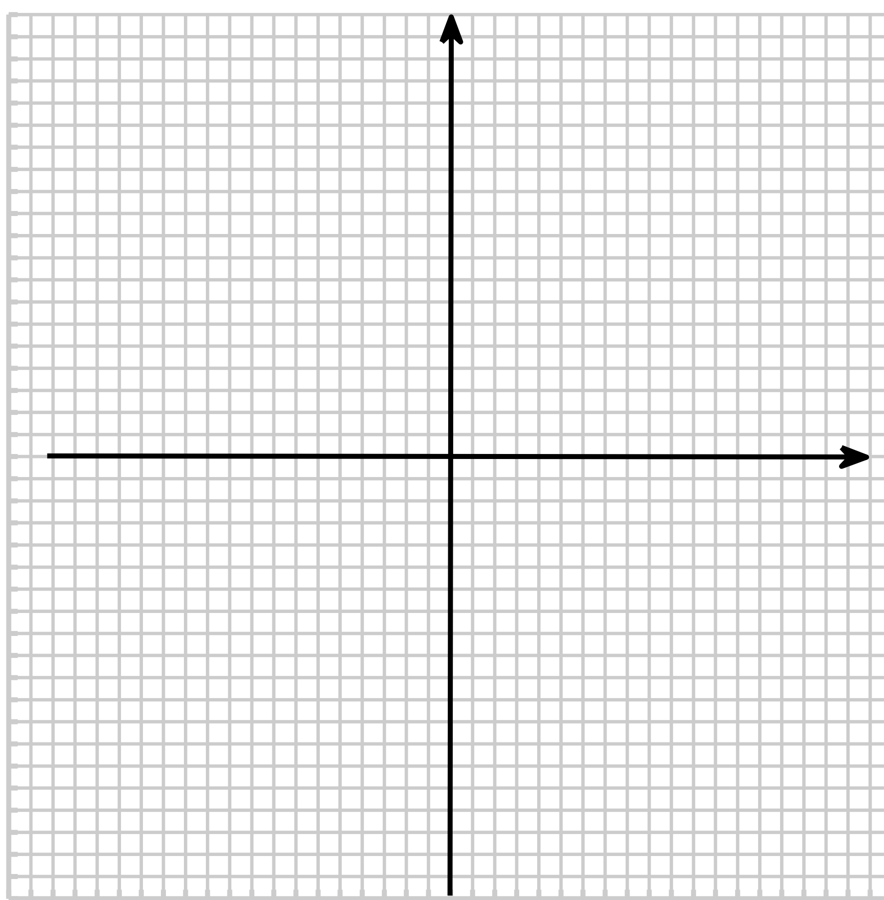
	n odd	n even
$a_n > 0$		
$a_n < 0$		

6. Begin graphing starting at the left 'end behavior point' that you marked your graph with. Proceed to the first zero on the left:
   - if the zero is odd, pass through the x axis,
   - if the zero is even 'bounce off' the x axis.
7. Continue to the next zero and repeat the process.
8. When you have finished this procedure with the last zero on the right, the graph should connect with the right end behavior arrow that you marked in step 4.
9. As a 'check point' you can determine the sign (+ or -) of the y intercept (when $x = 0$) and see if the result is consistent with your graph.

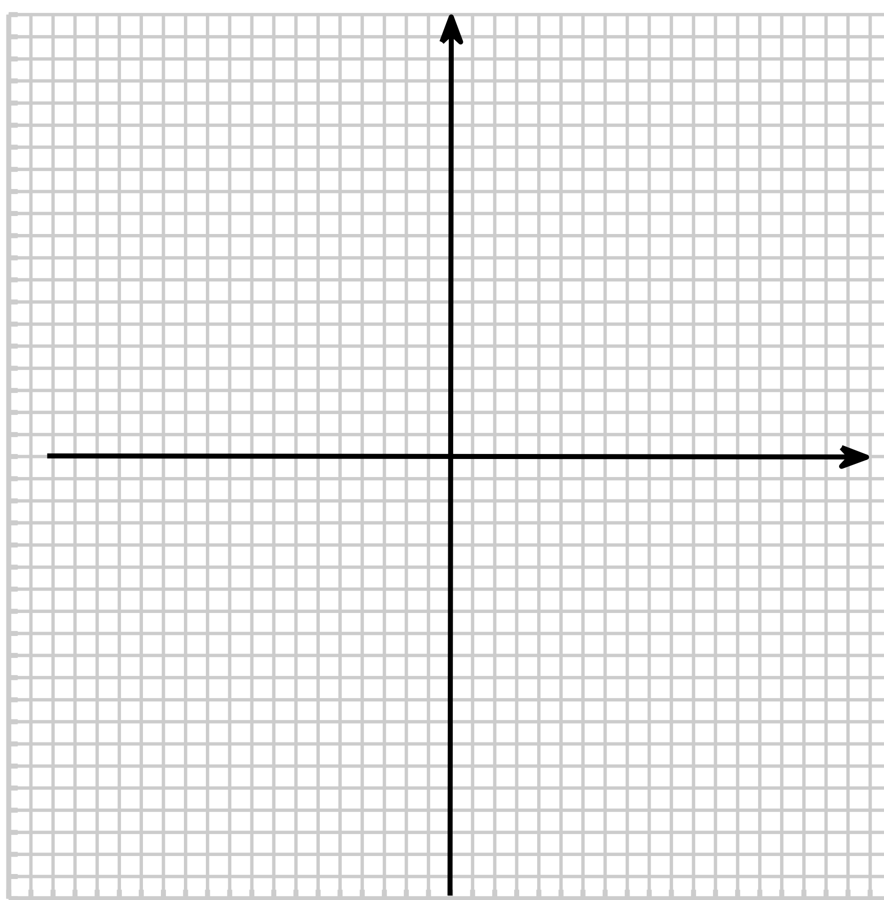
9. Plot $f(x) = x^5 - x^4 + 3x^3 - 3x^2 - 4x + 4$



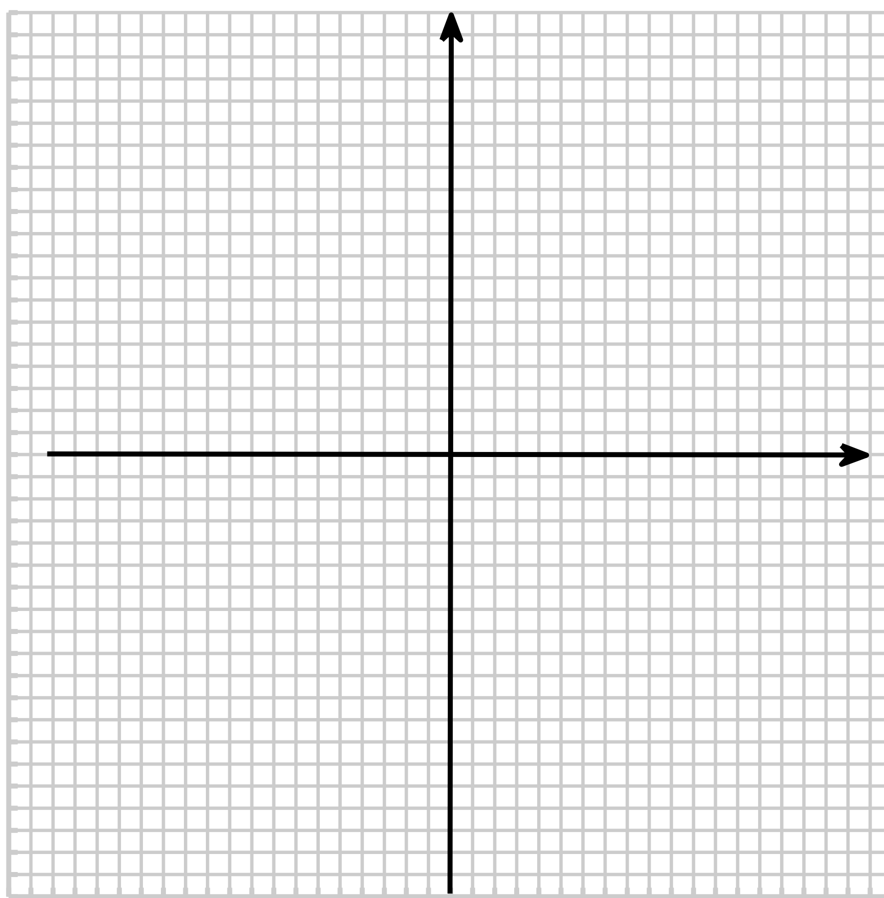
10. Plot $f(x) = x^3 - x^2 + 14x - 8$

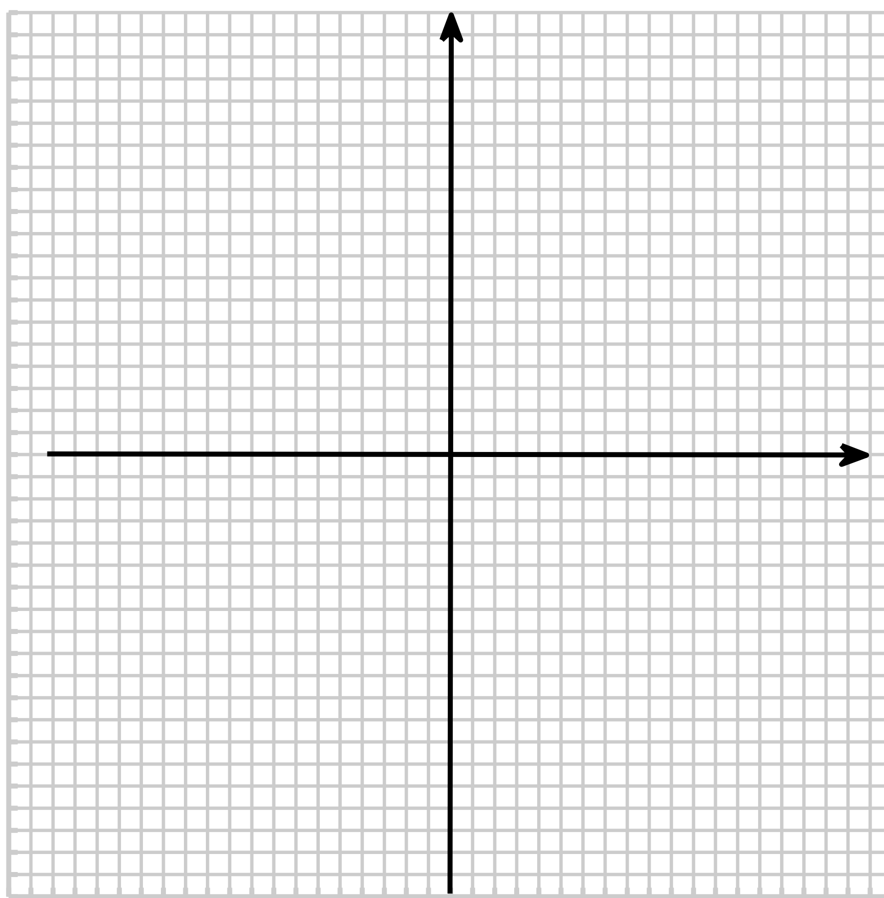


11. $f(x) = 2x^4 - x^3 - 19x^2 + 9x + 9$



12.





3.5 Complex Numbers

Name: ________________________

Definition 4. *A* ***complex number*** *is an expression of the form*

$$a + bi$$

where a and b are real numbers and $i^2 = -1$*. The* ***real part*** *of this complex number is a and the* ***imaginary part*** *is b. Two complex numbers are* **equal** *if and only if their real parts are equal and their imaginary parts are equal.*

Examples of complex numbers

$4 + 5i$	real part 4, imaginary part 4
$1 - i$	real part 1, imaginary part -1
$6i$	real part 0, imaginary part 6
-7	real part -7, imaginary part 0

Addition	**Description**
$(a + bi) + (c + di) = (a + c) + (b + d)i$	To add complex numbers, add the real parts and the imaginary parts.
Subtraction	
$(a + bi) - (c + di) = (a - c) + (b - d)i$	To subtract complex numbers, subtract the real parts and the imaginary parts.
Multiplication	
$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$	Multiply complex numbers like binomials, using $i^2 = -1$.

Express the following in the form $a + bi$**.**

1. $(5 + 2i) + (3 + 8i)$ 1. ____________
2. $(1 - 2i) - (5 - 3i)$ 2. ____________
3. $(1 - 2i)(5 - 3i)$ 3. ____________
4. $(7 + 3i)(4 + 12i)$ 4. ____________
5. i^9 5. ____________
6. i^{22} 6. ____________

Definition 5. *For a complex number $z = a + bi$, we define its **complex conjugate** to be $\bar{z} = a - bi$. Note that*

$$z \cdot \bar{z} = (a + bi)(a - bi) = a^2 + b^2$$

For example, the conjugate of $7 + 5i$, written as $\overline{7 + 5i}$, equals $7 - 5i$.

7. $\overline{1 - 2i}$ 7. ____________

8. $\overline{2 + 3i}$ 8. ____________

9. $\overline{-i}$ 9. ____________

10. $\overline{-2}$ 10. ____________

Definition 6. *Dividing Complex Numbers*

To simplify the quotient $\dfrac{a + bi}{c + di}$, multiply by 1 in the form of the denominator's conjugate divided by itself.

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \left(\frac{c - di}{c - di}\right) = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}$$

Divide.

11. $\dfrac{8 - 3i}{4 + i}$ 11. ____________

12. $\dfrac{5 + 2i}{8i}$ 12. ____________

Definition 7. *Square Roots of Negative Numbers If $-r$ is negative, then the positive (principal) square root of $-r$ is*

$$\sqrt{-r} = i\sqrt{r}$$

The two square roots of $-r$ are $i\sqrt{r}$ and $-i\sqrt{r}$

We usually write $i\sqrt{b}$ instead of $\sqrt{b}i$ to avoid confusion with $\sqrt{bi}$.

Simplify.

13. $\sqrt{-25}$ 13. ____________

14. $\sqrt{-5}$ 14. ____________

15. $-\sqrt{-169}$ 15. ____________

CAUTION

$\sqrt{a}\cdot\sqrt{b} = \sqrt{ab}$ when both a and b are positive, this is not true when both a and b are negative. For instance

$$\sqrt{-2}\cdot\sqrt{-3} = i\sqrt{2}\cdot i\sqrt{3} = i^2\sqrt{6} = -\sqrt{6}$$

but

$$\sqrt{(-2)\cdot(-3)} = \sqrt{6}$$

so

$$\sqrt{-2}\cdot\sqrt{-3} \neq \sqrt{(-2)\cdot(-3)}$$

Evaluate and express in the form $a + bi$.

16. $(\sqrt{12} - \sqrt{-3})(\sqrt{3} + \sqrt{-4})$ 16. ____________

17. $(\sqrt{3} - \sqrt{-5})(1 + \sqrt{-1})$ 17. ____________

We have already seen that if $a \neq 0$, the solutions of the quadratic equation $ax^2 + bx + c$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $b^2 - 4ac < 0$, then the equation has no real solution. But in the complex number system, this equation will always have solutions, because negative numbers have square roots in this expanded setting.

Solve each equation.

18. $9x^2 + 4 = 0$ 18. ____________

19. $x^2 + 2x + 2 = 0$ 19. ____________

20. $x^2 + \frac{1}{2}x + 1 = 0$ 20. ____________

Theory of Polynomials Summary

1. ***Definition***: A number c is called a ***ZERO*** (or a ***ROOT***) of the function P if $P(c) = 0$. The graph of $y = P(x)$ crosses the x-axis at $x = c$ precisely when c is a zero of P.

2. ***The Remainder Theorem***: If a polynomial $P(x)$ is divided by $x - c$, then the remainder is $P(c)$.

3. ***Factor theorem***: $x = c$ **is a zero** of a polynomial function $y = P(x)$ if and only if $x - c$ is a factor of $P(x)$

4. ***Rational Zero Theorem (RZT)***: If the polynomial $P(x) = a_n\,x^n + a_{n-1}\,x^{n-1} + \cdots + a_1\,x + a_0$ has integer coefficients, then every rational zero of $P(x)$ is of the form $\frac{p}{q}$, where p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n.

5. ***Definition***: If the factor $x - c$ occurs k times in the complete factorization of the polynomial $P(x)$, then c is called a root of $P(x) = 0$ with multiplicity k.

6. ***Example***: Determine the multiplicity of each root of the fifth degree polynomial function $P(x) = x^5 + x^4 - 6x^3 - 4x^2 + 8x$ given that P has the complete factorization $\boxed{P(x) = x(x+2)^2(x-1)(x-2).}$

factors of $P(x)$	associated roots	multiplicity	parity of multiplicity
x	$x = 0$	1	odd
$(x+2)$	$x = -2$	2	even
$(x-1)$	$x = 1$	1	odd
$(x-2)$	$x = 2$	1	odd

7. ***Intermediate Value Theorem (IVT)***: If $f(x)$ is a continuous function on the closed interval $[a, b]$ and k is a value between $f(a)$ and $f(b)$, then there exists at least one value c in the interval $[a, b]$ such that $f(c) = k$. Since all polynomial functions are continuous, if $f(a)$ and $f(b)$ have opposite signs, there must be a zero in the interval $[a, b]$

8. ***Fundamental Theorem of Algebra***: A polynomial function (or polynomial equation) with real coefficients of degree n has at most n distinct real roots, and exactly n (not necessarily distinct) complex roots.

9. ***Corollary***: If the polynomial P has real coefficients and if the complex number z is a zero of P, then its complex conjugate $\bar{z}$ is also a zero of P.

10. ***Corollary***: An n^{th} degree polynomial has n roots and can be factored into a product of n linear factors; i.e., $P(x) = (a_1x + b_1) \cdot (a_2x + b_2) \cdot (a_3x + b_3) \cdots (a_{n-1}x + b_{n-1}) \cdot (a_nx + b_n)$

11. ***Corollary***: A polynomial of even degree has a minimum of zero real roots. A polynomial of odd degree has at least one real root.

12. ***Clarification***: The solutions (roots) of a polynomial equation $P(x) = 0$ are precisely the zeros of a polynomial function $y = P(x)$. Hence, the aforementioned theorems concerning zeros of polynomial functions also apply to roots of polynomial equations.

3.6 Complex Zeros **Name: ________________________**

Find all the zeros of *P*, and write the complete factorization of *P*.

1. $P(x) = x^3 - 3x^2 + x - 3$

2. $P(x) = x^3 - 2x + 4$

3. $P(x) = 3x^5 + 24x^3 + 48x$

4. Find a polynomial $P(x)$ of degree 4, with zeros $i, -i, -2$ and 2, and with $P(3) = 25$.

5. Find a polynomial $P(x)$ of degree 4, with zeros $-2, 0$, where -2 is a zero of multiplicity 3.

6. Find all four zeros of $P(x) = 3x^4 - 2x^3 - x^2 - 12x - 4$.

7. Let $P(x) = x^4 + 2x^2 - 8$.
   (a) Factor P into linear and irreducible factors with real coefficients.
   (b) Factor P completely into linear factors with complex coefficients.

3.7 Rational Functions **Name:** ________________________

Definition 8. *A* *Rational Function* *is a function that has the form:*

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials. The domain of f is the set of all real numbers x, such that $q(x) \neq 0$.

Examples:

$$f(x) = \frac{1}{x} \qquad f(x) = \frac{2-x}{3x+1} \qquad f(x) = \frac{4x-3}{x^2+3x-4}$$

Is $f(x)$ a rational function?

1. $f(x) = \dfrac{x^3 - x - 3}{x^{1/2} + 3}$ 1. ____________

Write dom$\left(f(x)\right)$ using interval notation.

2. $f(x) = \dfrac{x-2}{x^2+7x}$ 2. ____________

3. $f(x) = \dfrac{1-x^2}{x+3}$ 3. ____________

4. $f(x) = \dfrac{x^2-4}{x-2}$ 4. ____________

Finding Intercepts

- Set $p(x) = 0$ and solve for x to find the x intercept(s).

- Evaluate $f(0)$ to find the y intercept.

Find the intercepts of $f(x)$.

5. $f(x) = \dfrac{x - 2}{x^2 - 7x + 6}$ 5. ____________

6. $f(x) = \dfrac{1 - x^2}{x + 3}$ 6. ____________

Solving rational inequalities. Find all values of x for which $f(x) > 0$.

7. $f(x) = \dfrac{x^2 + 2x - 8}{x^3 + 2x^2}$ 7. ____________

8. $f(x) = \dfrac{(x + 3)}{(x - 5)^2}$ 8. ____________

9. $f(x) = \dfrac{(x+4)^2(x-2)}{(x-3)^3(x+4)(x-5)^2}$ 9. ____________

10. Solve $\dfrac{x^2 + 3x - 4}{x^2 + 2x} \le 0$ 10. ____________

Identify the vertical asymptotes of $f(x)$, and write limit statements describing the behavior of the graph of f about its vertical asymptotes.

11. $f(x) = \dfrac{x^2 + 2x - 8}{x^3 + 2x^2}$ 11. ____________

12. $f(x) = \dfrac{(x+3)}{(x-5)^2}$ 12. ____________

13. $f(x) = \dfrac{(x+4)^2(x-2)}{(x-3)^3(x+4)(x-5)^2}$ 13. ____________

Theorem 1. *Let $f(x) = \dfrac{ax^n + ...}{bx^d + ...}$, where ax^n and bx^d are the leading terms of the polynomial in the numerator and denominator. Note that n is the degree of the numerator polynomial and d is the degree of the denominator polynomial.*

1. *if $n < d$, then the HA occurs at $y = 0$.*
2. *if $n = d$, then the HA occurs at $y = \dfrac{a}{b}$, the ratio of the leading coefficients.*
3. *if $n > d$, then the graph of $f(x)$ has a slant asymptote; found by using long division to divide the polynomial in the numerator by the polynomial in the denominator. y equals the quotient is the equation representing the slant asymptote.*

End Behavior *As $x \to \infty$ and as $x \to -\infty$, $f(x) \to c$, where $y = c$ is the HA. If the graph has a slant asymptote, then the end behavior of f is the same as the end behavior of the the polynomial quotient after doing the long division.*

Identify the horizontal asymptote of $f(x)$. Then write a description of the end behavior of the graph of f.

14. $f(x) = \dfrac{4}{4x + 3}$ 14. ____________

15. $f(x) = \dfrac{x + 4}{2x - 5}$ 15. ____________

16. $f(x) = \dfrac{x^2 - 3x + 1}{x^2 + 6x - 9}$ 16. ____________

17. $f(x) = \dfrac{x^2}{1 - x^3}$ 17. ____________

18. $f(x) = \dfrac{x^3 - 2x + 1}{x + 1}$ 18. ____________

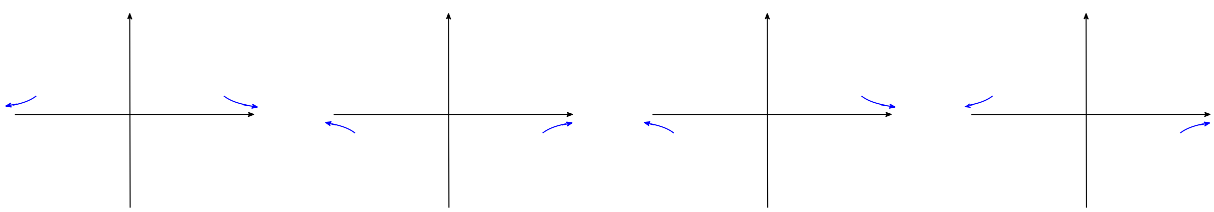
Graphing Rational Functions Algorithm

1. Write the numerator and denominator in factored form. Determine the locations of the x and y intercepts.
2. Determine the equation(s) that represent the Vertical Asymptote(s): set the denominator equal to zero and solve for x.
3. Determine the equation that represents the Horizontal or Slant Asymptote (HA or SA).
4. Write a statement describing the end behavior of the graph:

$$y \to HA \quad \text{as} \quad x \to \pm\infty \qquad \text{or} \qquad y \to SA \quad \text{as} \quad x \to \pm\infty$$

5. Determine the x intervals for which $f(x) < 0$ and $f(x) > 0$. Make a sign chart and use either the test point method or the multiplicity method to mark the sign (+ or -) of $f(x)$ for x in each interval.
6. Mark the "end behavior" of the graph, depending on whether $n < d$, $n = d$, or $n > d$; where n is the degree of the polynomial in the numerator and d is the degree of the polynomial in the denominator.

   Case 1: ($n < d$) The HA is $y = 0$. The end behavior of the graph will look like one of the following 4 pictures: Your inequality sign chart will indicate which one of the four



   you have; but you may have to compute a couple of y values be sure.

   Case 2: ($n = d$) The HA occurs at $y = \frac{a}{b}$. The end behavior will look like one of the four above graphs, but the HA is shifted up or down to $y = \frac{a}{b}$.

   Case 3: ($n > d$) The graph doesnt have a HA, it has a SA at $y = mx + b$, where $mx + b$ is the linear part of the resulting quotient, after the polynomial in the numerator has been divided by the polynomial in the denominator. The end behavior for the graph of the rational function will be the same as the end behavior of the SA. Draw the graph of $y = mx + b$ with a dotted line.
7. Identify and plot the zeros (also called roots) of the rational function.
   - Label these odd (O) or even (E) as in polynomial graphing
   - also plot the y intercept
8. Sketch the vertical asymptotes as vertical dotted lines: Label these VAs as being odd (O) or even (E) at the top of the dotted line depending on their multiplicities.
9. Determine the behavior of the graph (y value) around the VAs.

$$y \to \pm\infty \quad \text{as} \quad x \to a^- \quad \text{and} \quad y \to \pm\infty \quad \text{as} \quad x \to a^+$$

Whether or not y goes to plus or minus infinity depends on the sign of f(x) in that interval.

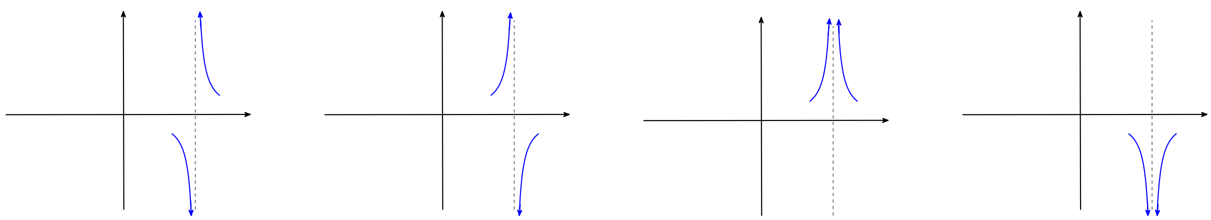
10. Begin graphing from the left end point and proceed to the first critical value
    - a critical value is a zero or a vertical asymptote
    - you may need to make a small table of values for better plotting accuracy.
11. At each zero proceed as in polynomial graphing, ie. If the zero is odd the graph goes thru the x axis; if the zero has even multiplicity, then it touches the axis and bounces off.
12. At each vertical asymptote, as you approach from the left go "north" to $+\infty$ or "south" to $-\infty$.

    At each **odd** vertical asymptote, as you step across the dotted line

    Switch signs from $+\infty$ or to $-\infty$ or from $-\infty$ or to $+\infty$

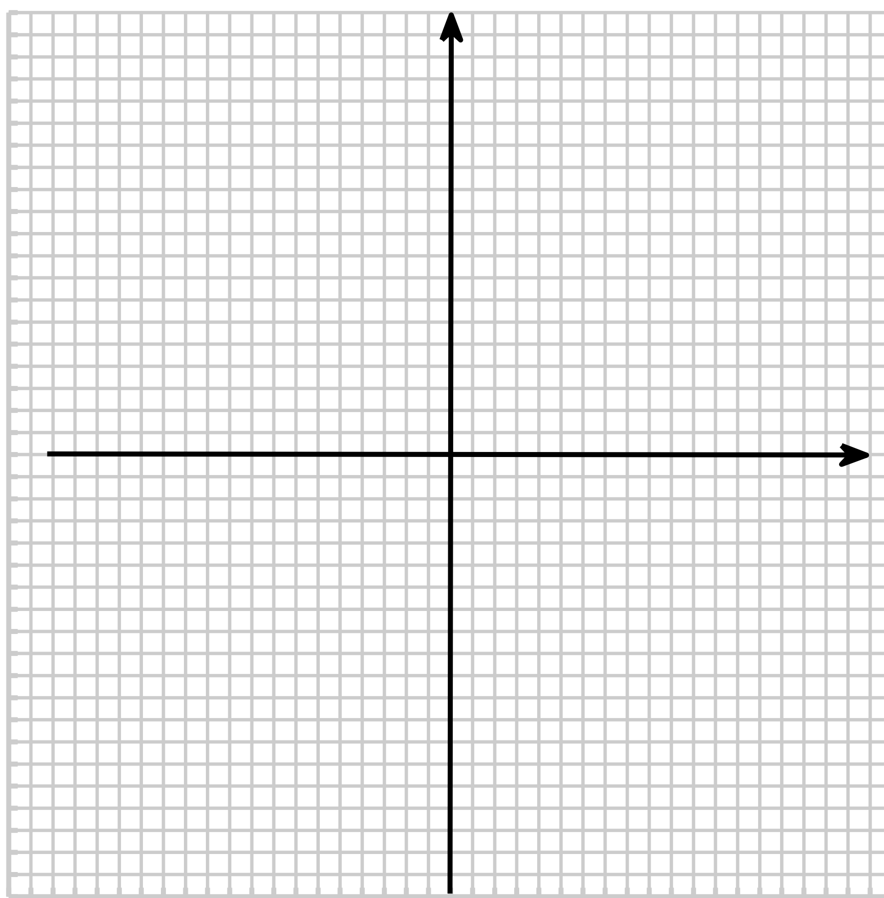
    At each **even** vertical asymptote, as you step across the dotted line

    Keep the **same signs** from $+\infty$ or to $+\infty$ or from $-\infty$ or to $-\infty$



13. Continue to the next critical value (zero or asymptote) repeat the process of step 8 or 9; ignore the vertical scale as you pass from one zero to the next; item When you have finished this procedure with the last zero on the right, the graph should connect with the right "endpoint" that you marked in step 4
14. As a "check point" you can determine the sign (+ or −) of the y intercept (when $x = 0$) and see if the result is consistent with your graph

19.



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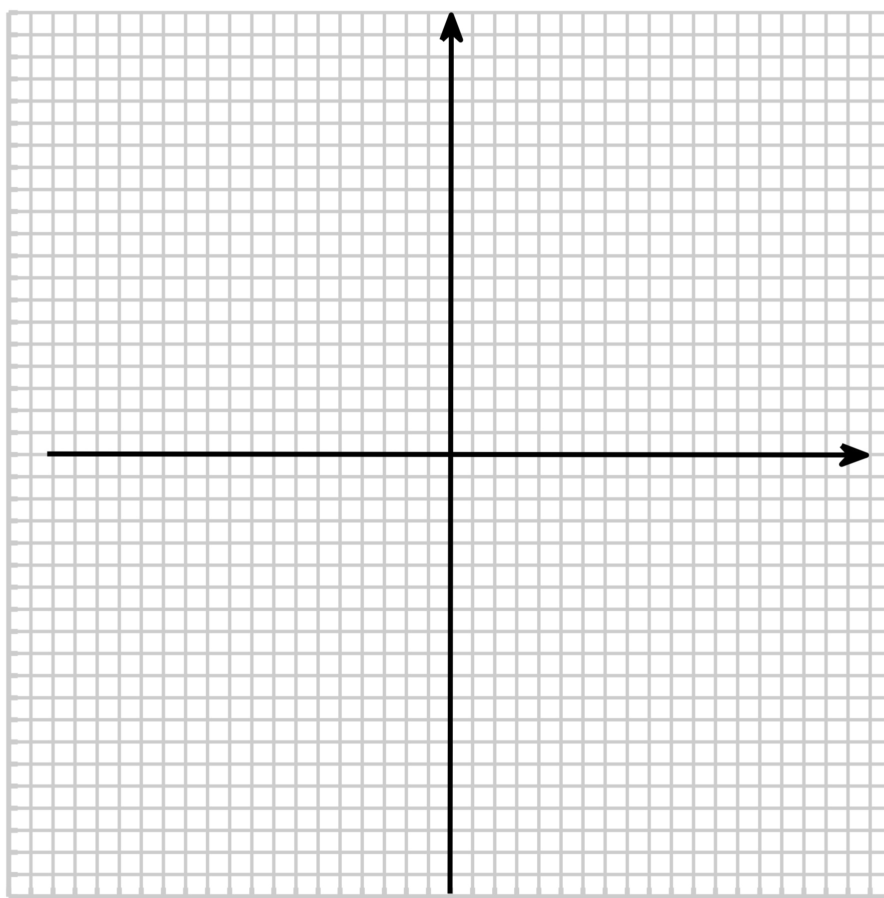

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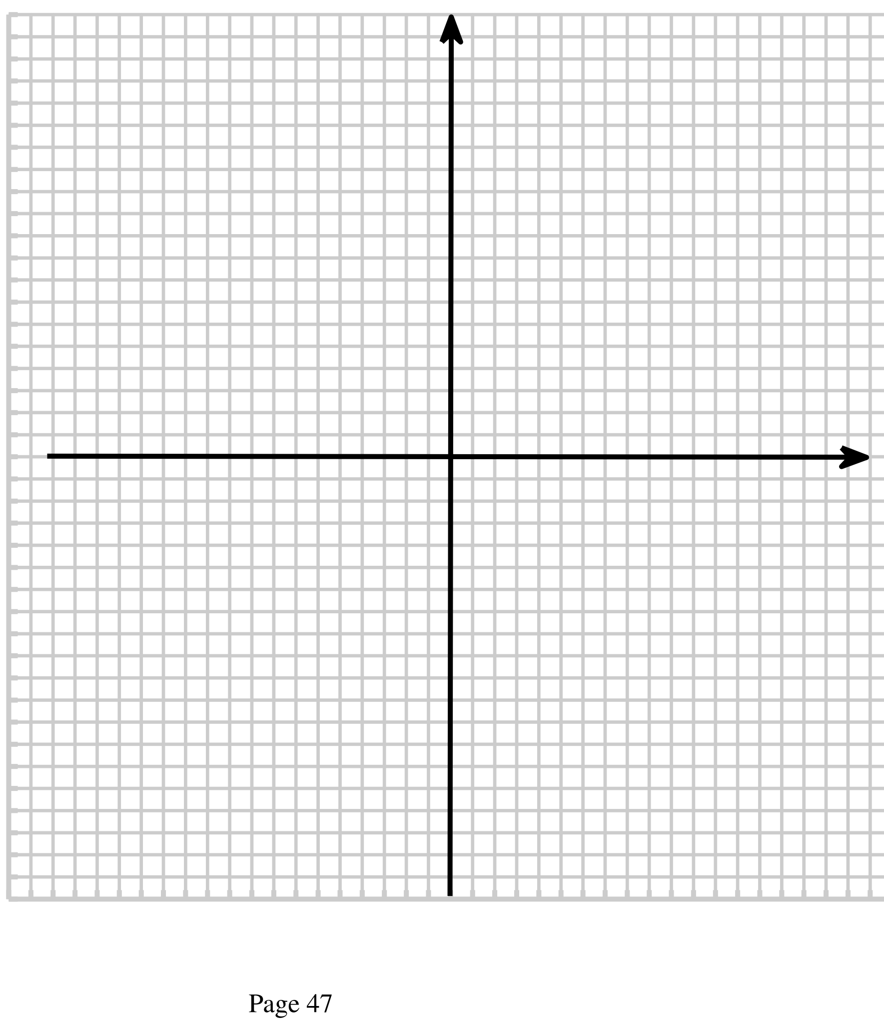


4.1 Exponential Functions **Name:** ______________________

Definition 9. *An exponential function with base a is a function of the form*

$$f(x) = a^x,$$

where a *and* x *are real numbers and*

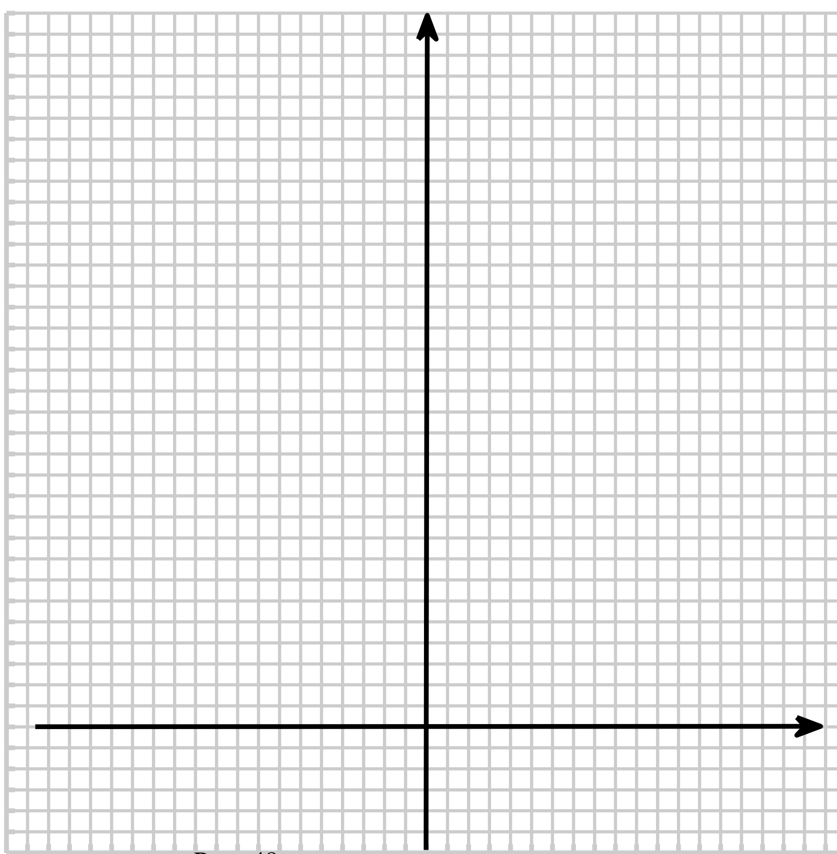
- *x is the independent VARIABLE of the function; and*
- *a is a number FIXED CONSTANT such that $a > 0$ and $a \neq 1$.*

Use a calculator to evaluate $f(x) = 7^x$ at the indicated values.

1. $f(-0.5)$, 1. ____________
2. $f(\sqrt{3})$, 2. ____________
3. $f(\frac{1}{6})$ 3. ____________
4. $f(x) = 3^{x-1}$; find $f(-4.5)$, 4. ____________

Sketch the graph of $f(x) = 2^x$ and $f(x) = 2^x$. Use the length of two squares as a single unit.

x	$f(x) = 2^x$
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	

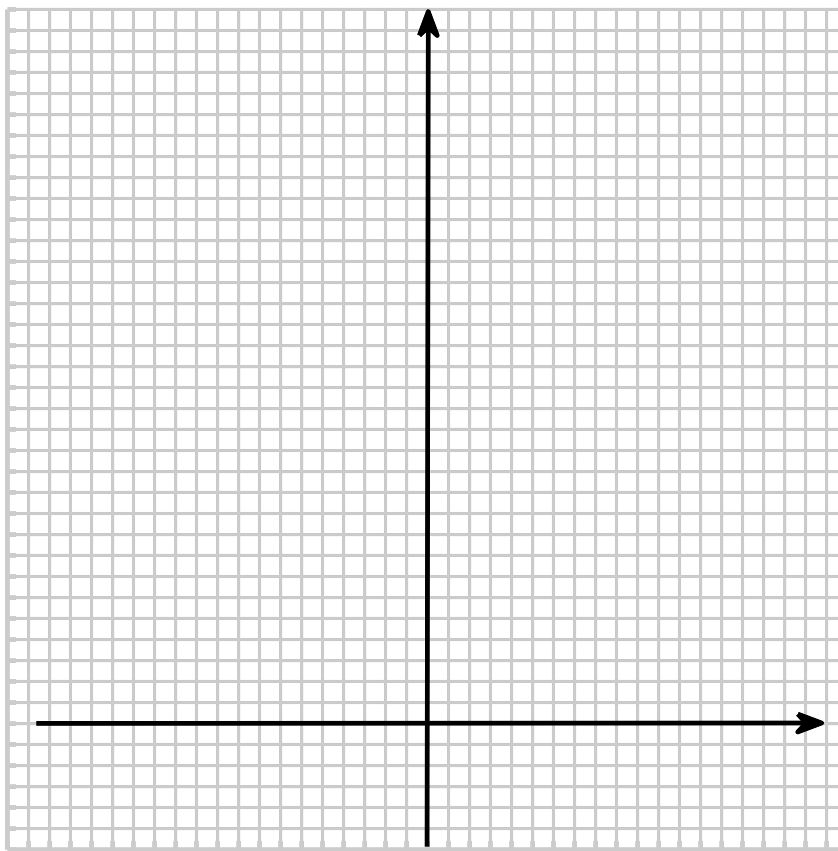


Let $g(x) = \left(\frac{1}{2}\right)^x$**. We know from transformations theory that the graph of** $g(x) = f(-x) = 2^{-x}$ **is obtained from reflecting the graph of** f **around the** y**-axis.**

Sketch the graph of $g(x)$ **on the same set of axes given above.**

Use the graph of $f(x) = 2^x$ **to sketch the graph of the given function.**

$g(x) = 2^{x-4} + 1$



5. What interval represents the domain of g? 5. ____________

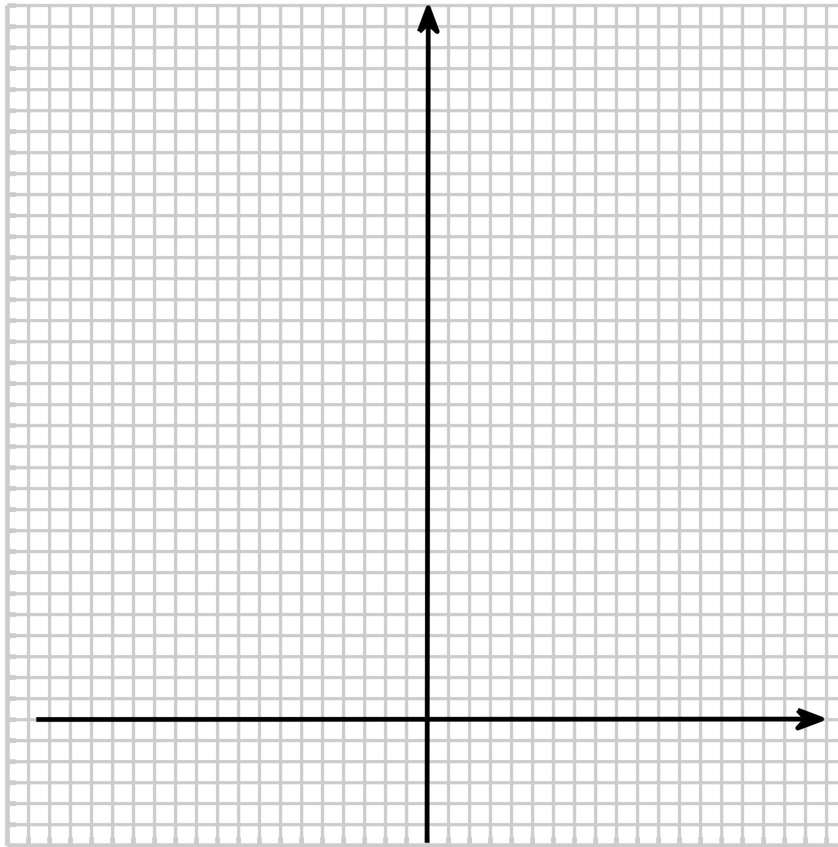
6. What equation represents the horizontal asymptote for g? 6. ____________

7. What interval represents the range of g? 7. ____________

8. What two statements describe the end behavior of the graph of the function? 8. ____________

Use the graph of $f(x) = 2^x$ to sketch the graph of the given function.

$g(x) = 2^{x-4} + 1$



9. What interval represents the domain of h? 9. ____________

10. What equation represents the horizontal asymptote for h? 10. ____________

11. What interval represents the range of h? 11. ____________

12. What two statements describe the end behavior of the graph of the function? 12. ____________

13. This is a ***Matching question*** associated with the theory on graphical translations of functions. Suppose $f(x) = 7^{-x}$. Relative to the graph of $f(x)$ the graphs of the following functions have been changed in what way?

_____ $g(x) = 7^{-x} + 5$ a.) shifted 5 units left

_____ $g(x) = 7^{(-x+5)}$ b.) reflected about the x axis

_____ $g(x) = -2 \cdot 7^{-x}$ c.) shifted 5 units down

_____ $g(x) = 7^{(-x-5)}$ d.) shifted 5 units right

_____ $g(x) = 7^{-x} - 5$ e.) shifted 5 units vertically up

14. This is a ***Matching question*** associated with the theory on graphical translations of functions. Suppose $f(x) = 5^x$. Relative to the graph of $f(x)$ the graphs of the following functions have been changed in what way?

_____ $g(x) = 5^x + 5$ a.) shifted 5 units down

_____ $g(x) = 5^{(x+5)}$ b.) reflected about the x axis

_____ $g(x) = -5^x$ c.) shifted 5 units up

_____ $g(x) = 5^{(x-5)}$ d.) shifted 5 units left

_____ $g(x) = 5^x - 5$ e.) shifted 5 units right

15. Use your graphing calculator to sketch the graphs of $f(x) = 2^x$ and $g(x) = x^2$. Which function is the dominant function? Compare the two functions in the viewing rectangles:

(a) [0,3] by [0,8]
(b) [0,6] by [0,25]
(c) [0,20] by [0,1000]

16. What interval represents the range of $y = -3^{-x} - 4$? 16. ____________

Exponential functions occur in calculating compound interest. If an amount of money P, called the ______________, is invested at an interest rate i per time period, then after one time period the interest is ______, and the amount of money, A, in the account is ________________________. If the interest is reinvested, then the new principal is __________________, and the amount after another time period is $A =$ ____________

Similarly, after a third time period the amount is ____________. In general, after k periods the amount is __________________. Note that this is an exponential function with base ____________.

If the annual interest rate is r and if interest is compounded n times per year, then in each time period the interest rate is $i = r/n$, and there are nt time periods in t years. This leads to the following formula for the amount after t years.

Compound Interest Formula
If a principal P (dollars) is invested for t years at an annual rate r, and it is compounded n times per year, then the amount A, or ending balance, is given by

$$A = P\left(1 + \frac{r}{n}\right)^{n \cdot t}$$

17. Suppose you invest \$1000 at an interest rate of 7% per year. Find the amounts in the account after 3 years if interest is compounded annually, semiannually, quarterly, monthly, and daily.

Compounding	n	Amount after 3 years
Annual	1	
Semiannual		
Quarterly		
Monthly		
Daily		

18. A certain breed of mouse was introduced onto a small island with an initial population of 320 mice and scientists estimate that the mouse population is doubling every year.

(a) Find a function that models the number of mice after t years.

(b) Estimate the mouse population after 8 years.

4.2 The Natural Exponential Function **Name:** ______________________

Definition 10. *The number e is defined as the value that* $A(n) = \left(1 + \frac{1}{n}\right)^n$ *approaches as n becomes large.*

n	$A(n) = \left(1 + \frac{1}{n}\right)^n$
1	2.00
2	2.25
3	2.370370
4	2.441406
5	2.488320
100	2.704814
365	2.714567
8760	2.718127

Definition 11. *The natural exponential function is the exponential function*

$$f(x) = e^x$$

with base e.

1. What is the range of $f(x) = -e^{-x} + 3$? What equation represents the horizontal asymptote for $f(x) = -e^{-x} + 3$?

1. ____________

2. What is the range of $f(x) = -2e^{x-2} - 4$? What equation represents the horizontal asymptote for $f(x) = -2e^{x-2} - 4$?

2. ____________

In the last section, we saw that interest paid increases as the number of compounding periods n increases. Suppose we let n grow large enough that the interest would be continuously added to the amount in the account. If we let $m = \frac{n}{r}$, then

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} = P\left(1 + \frac{1}{\frac{n}{r}}\right)^{n \cdot t \cdot \frac{r}{r}} = P\left[\left(1 + \frac{1}{\frac{n}{r}}\right)^{\frac{n}{r}}\right]^{rt} = P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt} = Pe^{rt}$$

This gives the amount when the interest is compounded at "every instant."

Continuously compounded interest is calculated by the formula

$$A(t) = Pe^{rt}$$

where

$A(t)$ = amount after t years
P = principal
r = interest rate per year
t = number of years.

3. Suppose you invest \$1000 at an interest rate of 7% per year. Find the amounts in the account after 3 years if interest is continuously compounded.

3. ____________

4. Doctors use radioactive iodine as a tracer in diagnosing certain thyroid gland disorders. This type of iodine decays in such a way that the mass remaining after t days is given by the function

$$m(t) = 6e^{-0.087t}$$

where $m(t)$ is measured in grams.

(a) Find the mass at time $t = 0$.

(b) How much of the mass remains after 20 days?

4. ____________

4.3 Logarithmic Functions **Name:** ________________________

Definition 12. *Let $a > 0$ and $a \neq 1$. Then $\log_a(x)$ is the exponent we raise a to get x.*

1. Write, in words, the meaning of $\log_5(25)$.

2. What number does $\log_5(25)$ represent? 2. ____________

3. Write, in words, the meaning of $\log_7(1)$.

4. What number does $\log_7(1)$ represent? 4. ____________

5. Write, in words, the meaning of $\log_{\frac{1}{2}}(16)$.

6. What number does $\log_{\frac{1}{2}}(16)$ represent? 6. ____________

Definition 13. *Let a be a positive number with $a \neq 1$. The logarithm function with base a, denoted* ***log****$_a$, is defined by*

$$\left[\log_a x = y\right] \iff \left[a^y = x\right]$$

So $\log_a(x)$ is the exponent we raise a to get x.

Write each in logarithmic form.

7. $2^4 = 16$ 7. ____________

8. $10^4 = 10,000$ 8. ____________

9. $0.001 = 10^3$ 9. ____________

10. $\left(\frac{1}{3}\right)^{-2} = 9$ 10. ____________

11. $10^{x+5} = 10$ 11. ____________

Write each in exponential form.

12. $\log_3\left(\frac{1}{81}\right) = -4$ 12. ______________

13. $\log_7 49 = 2$ 13. ______________

14. $\log_5 125 = 3$ 14. ______________

15. $\log_4 x = \frac{1}{2}$ 15. ______________

16. $\log_x 9 = 2$ 16. ______________

Properties of Logarithms

Property	**Reason**
1. $\log_a 1 = 0$	We must raise a to the power 0 to get 1.
2. $\log_a a = 1$	We must raise a to the power 1 to get a.
3. $\log_a a^x = x$	We must raise a to the power x to get a^x.
4. $a^{\log_a x} = x$	$\log_a x$ is the power to which a must be raised to get x.

Evaluate Each Expression.

17. $\log_3 1$ 17. ______________

18. $\log_4 64$ 18. ______________

19. $\log_8 8^{17}$ 19. ______________

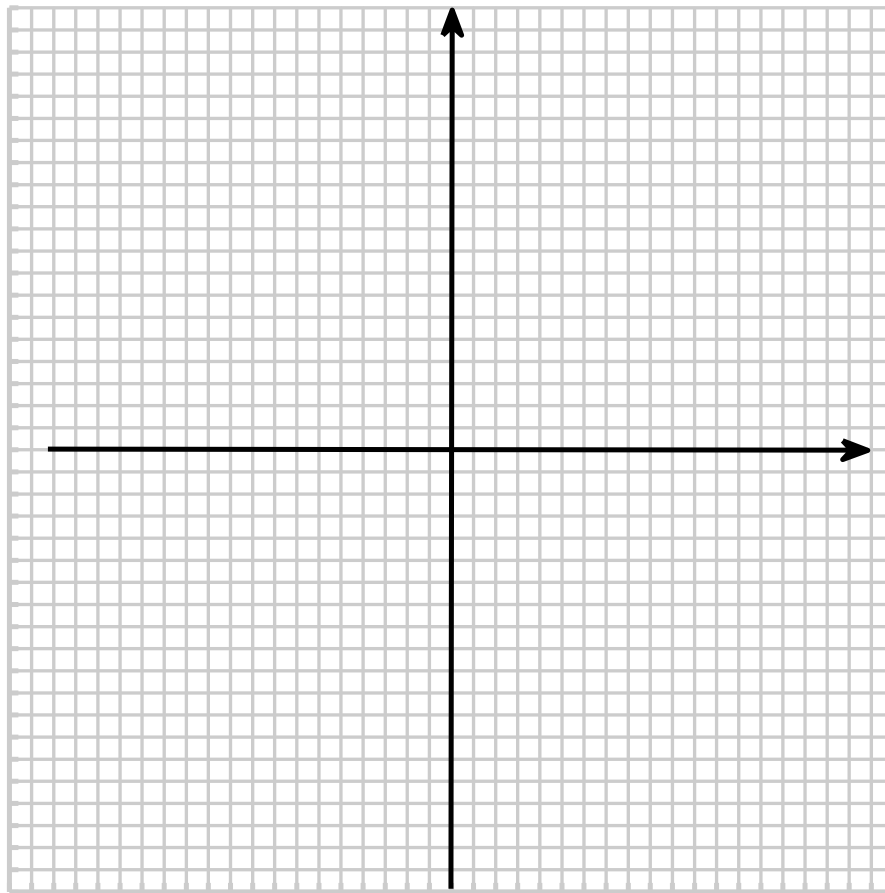
20. $\log_{10} \sqrt{10}$ 20. ______________

21. $3^{\log_3 8}$ 21. ______________

22. $10^{\log 5}$ 22. ______________

Sketch the graphs of $f(x) = 2^x$ and $f^{-1}(x) = \log_2(x)$ on the same set of axes.

x	2^x
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	



23. What interval represents the domain of $f^{-1}(x)$? 23. ____________

24. What equation represents the vertical asymptote for $f^{-1}(x)$? 24. ____________

25. What interval represents the range of $f^{-1}(x)$? 25. ____________

26. What two statements describe the end behavior of the graph of $f^{-1}(x)$? 26. ____________

27. This is a ***Matching question*** associated with the theory on graphical translations of functions. Suppose $f(x) = \log_2 x$. Relative to the graph of $f(x)$ the graphs of the following functions have been changed in what way?

_____ $g(x) = \log_2 x + 5$ a.) shifted 5 units left

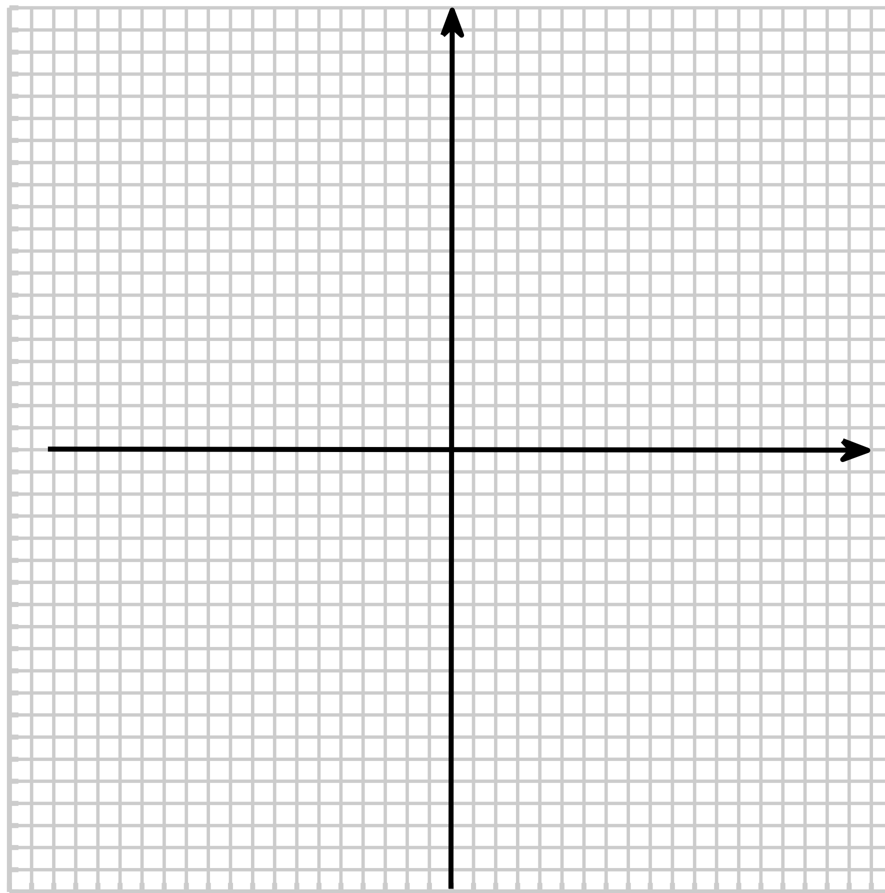
_____ $g(x) = \log_2(x + 5)$ b.) reflected about the x axis

_____ $g(x) = -2 \cdot \log_2(x)$ c.) shifted 5 units down

_____ $g(x) = \log_2(x - 5)$ d.) shifted 5 units right

_____ $g(x) = \log_2(x) - 5$ e.) shifted 5 units vertically up

Use translations theory to graph $g(x) = -2\log(x+5)+1$**.**



28. What interval represents the domain of $g(x)$? 28. ____________

29. What equation represents the vertical asymptote for $g(x)$? 29. ____________

30. What interval represents the range of $g(x)$? 30. ____________

31. What two statements describe the end behavior of the graph of $g(x)$? 31. ____________

Definition 14. *The logarithm with base 10 is called the* ***common logarithm*** *and is described by omitting the base:*

$$\log(x) = \log_{10}(x)$$

Definition 15. *The logarithm with base e is called the* ***natural logarithm*** *and is denoted by* ***ln****:*

$$\ln(x) = \log_e(x)$$

Properties of Natural Logarithms

Property	**Reason**
1. $\ln 1 = 0$	We must raise e to the power 0 to get 1.
2. $\ln e = 1$	We must raise e to the power 1 to get e.
3. $\ln e^x = x$	We must raise e to the power x to get e^x.
4. $e^{\ln x} = x$	$\ln x$ is the power to which e must be raised to get x.

Evaluate Each Expression.

32. $\ln e^{-3}$ 32. ______________

33. $\ln\left(\frac{1}{e^4}\right)$ 33. ______________

34. $\ln 5$ 34. ______________

Find the domain of each function.

35. $f(x) = \log_2(4 - x^2)$ 35. ______________

36. $f(x) = \ln(x + 2)$ 36. ______________

37. $f(x) = \log_7(1 - 2x) + 4$ 37. ______________

4.4 Laws of Logs **Name:** ________________________

Laws of Logarithms
Let a be a positive number, with $a \neq 1$. Let A, B and C be any real numbers with $A > 0$ and $B > 0$.

1. $\log_a(AB) = \log_a A + \log_a B$

2. $\log_a\left(\dfrac{A}{B}\right) = \log_a A - \log_a B$

3. $\log_a(A^C) = C \cdot \log_a A$

1. **Evaluate each expression.**

(a) $\log_4 2 + \log_4 32$

(b) $\log_2 80 + \log_2 5$

(c) $-\dfrac{1}{3}\log 8$

2. **Use the Laws of Logarithms to expand each expression.**

(a) $\log_2(6x) + \log_5(x^3y^6)$

(b) $\log_5(x^3y^6)$

(c) $\ln\left(\dfrac{ab}{\sqrt[3]{c}}\right)$

3. Combine $4\log x + \dfrac{1}{2}\log(x+2)$ into a single logarithm.

4. Combine $3\ln x + \dfrac{1}{2}\ln y - 5\ln(x^2+2)$ into a single logarithm.

Change of Base Formula

$$\log_d n = \frac{\log_a n}{\log_a d} = \frac{\ln n}{\ln d} = \frac{\log n}{\log d}$$

Use the Change of Base Formula and a calculator to evaluate the logarithm, rounded to six decimal places.

5. $\log_3 6$ 5. ____________

6. $\log_7 48$ 6. ____________

7. $\log_{15} 97$ 7. ____________

4.5 Exp. & Log Eqns **Name: ________________________**

Solve for x, rounded to six decimal places.

1. $4^{2x-1} = 7$ 1. ____________

2. $8e^{3x} = 17$ 2. ____________

3. $e^{3-2x} = 4$ 3. ____________

Solve for x.

4. $e^{2x} - 4e^{x} = 5$

4. ____________

5. $3xe^{x} + x^{2}e^{x} = 0$

5. ____________

6. $\log_{2}(x + 4) = 3$

6. ____________

7. $\ln x = 3$ 7. ____________

8. $\log_3(1 - 2x) = 3$ 8. ____________

9. $4 + 3\log(2x) = 16$ 9. ____________

10. $\log(x + 2) + \log(x - 1) = 1$ 10. ____________

Use your graphing calculator to solve for x.

11. $4 + 3\log(2x) = 16$ 11. ____________

12. Environmental scientists measure the intensity of light at various depths in a lake to find the "transparency" of the water. Certain levels of transparency are required for the biodiversity of the submerged macrophyte population. In a certain lake the intensity of light at depth x is given by

$$I = 10e^{-0.008x}$$

where I is measured in lumens and x in feet.

(a) Find the intensity I at a depth of 30ft.

(b) At what depth has the light intensity dropped to $I = 5$?

Compound Interest

If a principal P is invested at an interest rate r for a period of t years, then the amount A of the investment is given by

$A = P(1 + r)$	Simple interest (for one year)
$A(t) = P\left(1 + \dfrac{r}{n}\right)^{nt}$	Interest compounded n times per year
$A(t) = Pe^{rt}$	Interest compounded continuously

13. A sum of \$1500 is invested at an interest rate of 8% per year. Find the time required for the money to double if the interest is compounded according to the following method.

(a) Semiannually

(b) Continuously

4.6 Applications **Name:** ________________________

Exponential Growth (Doubling Time) If the initial size of a population is n_0 and the doubling time is a, then the size of the population at time t is

$$n(t) = n_0 2^{t/a}$$

where a and t are measured in the same time units (minutes, hours, days, years, and so on).

1. Under ideal conditions a certain bacteria population doubles every three hours. Initially there are 200 bacteria in a colony.

   (a) Find a model for the bacteria population after t hours.

   (b) How many bacteria are in the colony after 15 hours?

   (c) When will the bacteria count reach 100,000?

2. A certain breed of rabbit was introduced onto a small island 8 months ago. The current rabbit population on the island is estimated to be 4100 and doubling every 3 months.

(a) What was the initial size of the rabbit population?

(b) Estimate the population one year after the rabbits were introduced to the island.

(c) Sketch a graph of the rabbit population.

Exponential Growth (Relative Growth Rate) A population that experiences exponential growth increases according to the model

$$n(t) = n_0 e^{rt}$$

where
- $n(t)$ = population at time t
- n_0 = initial size of the population
- r = relative rate of growth (expressed as a proportion of the population)
- t = time

3. The population of a certain species of fish has a relative growth rate of 1.2% per year. It is estimated that the population in 2000 was 12 million.

   (a) Find an exponential model $n(t) = n_0 e^{rt}$ for the population t years after 2000.

   (b) Estimate the fish population in the year 2005.

   (c) Sketch a graph of the fish population.

4. It is observed that a certain bacteria culture has a relative growth rate of 12% per hour, but in the presence of an antibiotic the relative growth rat is reduced to 5% per hour. The initial amount of bacteria in the culture is 22. Find the projected population after 24 hours for the following conditions.

(a) No antibiotic is present, so the relative growth rate is 12%.

(b) An antibiotic is present in the culture, so the relative growth rate is reduced to 5%.

5. The bat population in a certain Midwestern county was 350,000 in 2009, and the observed doubling time for the population is 25 years.

(a) Find an exponential model $n(t) = n_0 2^{t/a}$ for the population t years after 2009.

(b) Find an exponential model $n(t) = n_0 e^{rt}$ for the population t years after 2009.

(c) Estimate when the population will reach 2 million.

Radioactive Decay Model If m_0 is the initial mass of a radioactive substance with half-life h, then the mass remaining at time t is modeled by the function

$$m(t) = m_0 e^{-rt}$$

where $r = \dfrac{\ln 2}{h}$

6. The half-life of cesium-137 is 30 years. Suppose we have a 10 gram sample.

   (a) Find a function $m(t) = m_0 2^{-t/h}$ that models the mass remaining after t years.

   (b) Find a function $m(t) = m_0 e^{-rt}$ that models the mass remaining after t years.

   (c) How much of the sample will remain after 80 years?

   (d) After how long will only 2 grams of the sample remain?

Newton's Law of Cooling If D_0 is the initial temperature difference between an object and its surroundings, and if its surroundings have temperature T_s, then the temperature of the object at time t is modeled by the function

$$T(t) = T_s + D_0 e^{-kt}$$

where	$n(t)$	= population at time t
	n_0	= initial size of the population
	k	= relative rate of growth (expressed as a proportion of the population)
	t	= time

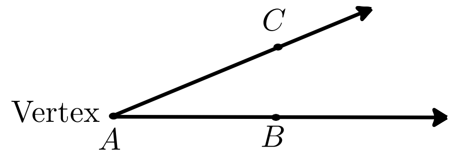
7. Newton's Law of Cooling is used in homicide investigations to determine the time of death. The normal body temperature is 98.6°F. Immediately following death, the body begins to cool. It has been determined experimentally that the constant in Newton's Law of Cooling is approximately $k = 0.1947$, assuming that time is measured in hours. Suppose that the temperature of the surroundings is 60°F.

   (a) Find a function $T(t)$ that models the temperature t hours after death.

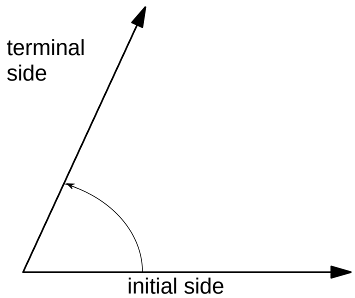
   (b) If the temperature of the body is now 72°F, how long ago was the time of death.

6.1 Angle Measure **Name:** ____________________

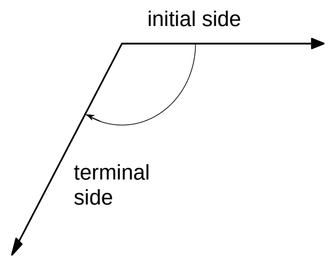
A ray is the set of points which are part of a line which is finite in one direction, but infinite in the other. An angle is defined as the union of two rays with a common endpoint, the vertex.



An angle can be thought of as being formed by rotating one ray away from a fixed ray. The fixed ray is the initial side and the rotated ray is the terminal side. If the rotation is counterclockwise the measure of the angle is considered positive, and if the rotation is clockwise the measure of the angle is considered to be negative.

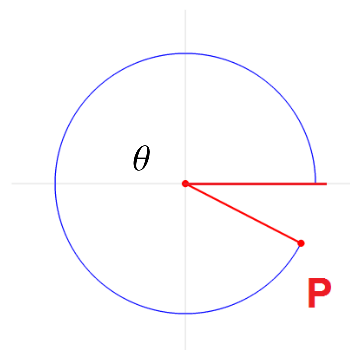


Positive Angle



Negative Angle

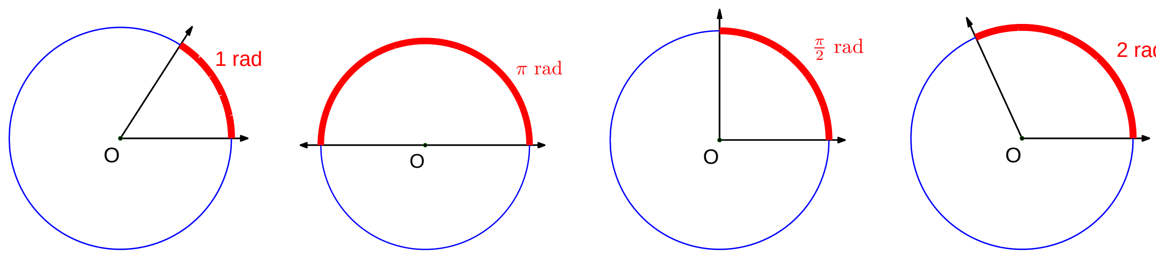
The measure of an angle is the amount of rotation it takes to rotate the terminal side away from the initial side. Let θ (read "theta") be an angle with a point P (not on the vertex) on the terminal side. As the terminal side of the angle is rotated through one revolution, the trace of the point P forms a circle. (animation link) But we often work with angles whose measure are less than one revolution.



Degree Measure
One complete revolution of a circle is divided into 360 equal parts called degrees. A **degree** (in full, a degree of arc, arc degree, or arcdegree), usually denoted by ° (the degree symbol), is a measurement of an angle, representing 1/360 of a full rotation.

$$360 \text{ degrees} = 360^\circ = 1 \text{ complete revolution}$$
$$1 \text{ degree} = 1^\circ = \frac{1}{360} \text{ of a complete revolution}$$

The selection of 360 as the number of divisions is rooted in history and may reflect the fact that the Earth completes one revolution about the Sun in approximately 365 days. Because 365 does not have a large number of divisors (only 5 and 73), the inventors of the system probably picked 360 with its multitude of divisors (2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72,



90, 120, and 180) as being close to the number of days of the year and easy to use in computing. The fortunate choice of 360 gives us a large number of even-degree angles that are simple fractions of a full revolution. We can also measure angles in radians.

Radian Measure

A radian (denoted rads) is the standard unit of angular measure, defined by the following ratio.

$$\text{radian measure of a given angle} \quad = \quad \frac{\text{arc length of the given angle}}{\text{radius length}}$$

Example: Suppose we have an angle whose associated arc length is $20m$. Suppose also that the vertex of the angle is centered inside a circle whose circumference is $400m$. Find the radian measure of this angle.

Did you notice that the units canceled or divided out of the quotient? This is because degrees and radians are dimensionless quantities. In dimensional analysis, a dimensionless quantity is a quantity without an associated physical dimension. It is thus a "pure" number, and as such always has a dimension of 1. Dimensionless quantities are widely used in mathematics, physics, engineering, economics, and in everyday life (such as in counting). Numerous well-known quantities, such as π, e, and ϕ (the Golden Ratio), are dimensionless. A larger list of dimensionless quantities is located here.

Dimensionless quantities are often defined as products or ratios of quantities that are not dimensionless, but whose dimensions cancel out when their powers are multiplied. This is the case, for instance, with degrees and radians. Even though a dimensionless quantity has no physical dimension associated with it, it can still have **dimensionless units**, for example, degrees, radians, ppm (parts per million), dozens, etc. Such notation does not indicate the presence of physical dimensions, and is purely a notational convention. Because we measure angles in two ways, we still need to name the unit, radians or degrees, rather than leaving it out entirely; but this is with the understanding that radians and degrees are dimensionless units.

When measuring in degrees, we will always say *degrees*, or use the degree symbol, as with $\theta = 22.5°$. When measuring in radians, we omit the word *radian*. That is, when no unit is given, radian measure is understood. For example, $\theta = 22.5$ means the measure of θ is 22.5 (radial lengths) radians.

One revolution is measured by 360° or 2π radians. Therefore,

$$\text{Number of revolutions} = \frac{\text{Angle in degrees}}{360}$$

$$\text{Number of revolutions} = \frac{\text{Angle in radians}}{2\pi}$$

Setting these two ratios equal to each other gives us a relationship between degree and radian measure.

$$\frac{\text{Angle in degrees}}{360} = \frac{\text{Angle in radians}}{2\pi}$$

Conversion between Radians and Degrees

$$180^\circ = \pi \text{ rad}$$

1. To convert degrees to radians, multiply by $\frac{\pi}{180}$
2. To convert radians to degrees, multiply by $\frac{180}{\pi}$

Find the radian measure of the angle with the given degree measure.

1. 84°   1. ____________

2. -75°   2. ____________

3. 3960°   3. ____________

Find the degree measure of the angle with the given radian measure. When no unit is given, the angle is assumed to be measured in radians.

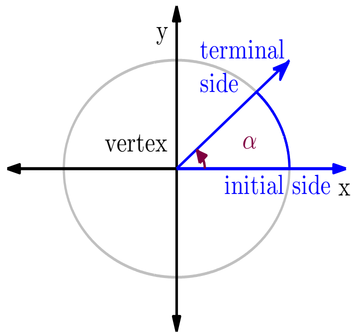
4. 1   4. ____________

5. $\frac{5\pi}{6}$ 5. ____________

6. $\frac{11\pi}{3}$ 6. ____________

7. 5   7. ____________

Definition 16. *An angle in* **standard position** *is located in a rectangular coordinate system with the vertex at the origin and the initial side on the positive x-axis.*



Draw the given angle in standard position.

8. $\theta = \dfrac{\pi}{6}$

9. $\theta = -120°$

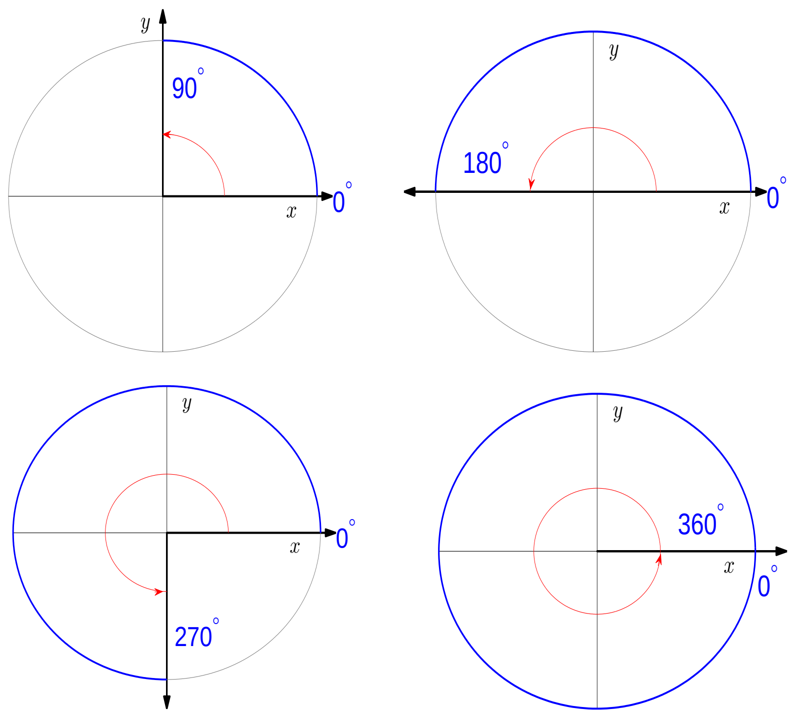
10. $\theta = 540°$

12. $\theta = \dfrac{\pi}{3}$

11. $\theta = \dfrac{5\pi}{3}$

13. $\theta = \dfrac{7\pi}{6}$

Definition 17. *Quadrantal angles are angles whose terminal side falls along an axis.*



An angle whose measure is a multiple of either 30^o, 45^o, 60^o or 90^o is called a **common angle**. We work with integer multiples of common angles a lot in this class.

Multiples of	are called	Examples (assuming angles are in reduced form)
$\frac{\pi}{2}$	90^o types quadrantal angles	$\frac{3\pi}{2}, \frac{11\pi}{2}\ -\frac{7\pi}{2}, \pi$
$\frac{\pi}{3}$	60^o types	$\frac{2\pi}{3}, \frac{5\pi}{3}\ -\frac{4\pi}{3}, -\frac{217\pi}{3}$
$\frac{\pi}{4}$	45^o types	$\frac{3\pi}{4}, \frac{5\pi}{4}\ -\frac{7\pi}{4}, -\frac{217\pi}{4}$
$\frac{\pi}{6}$	30^o types	$\frac{5\pi}{6}, \frac{7\pi}{6}\ -\frac{11\pi}{6}, -\frac{217\pi}{6}$

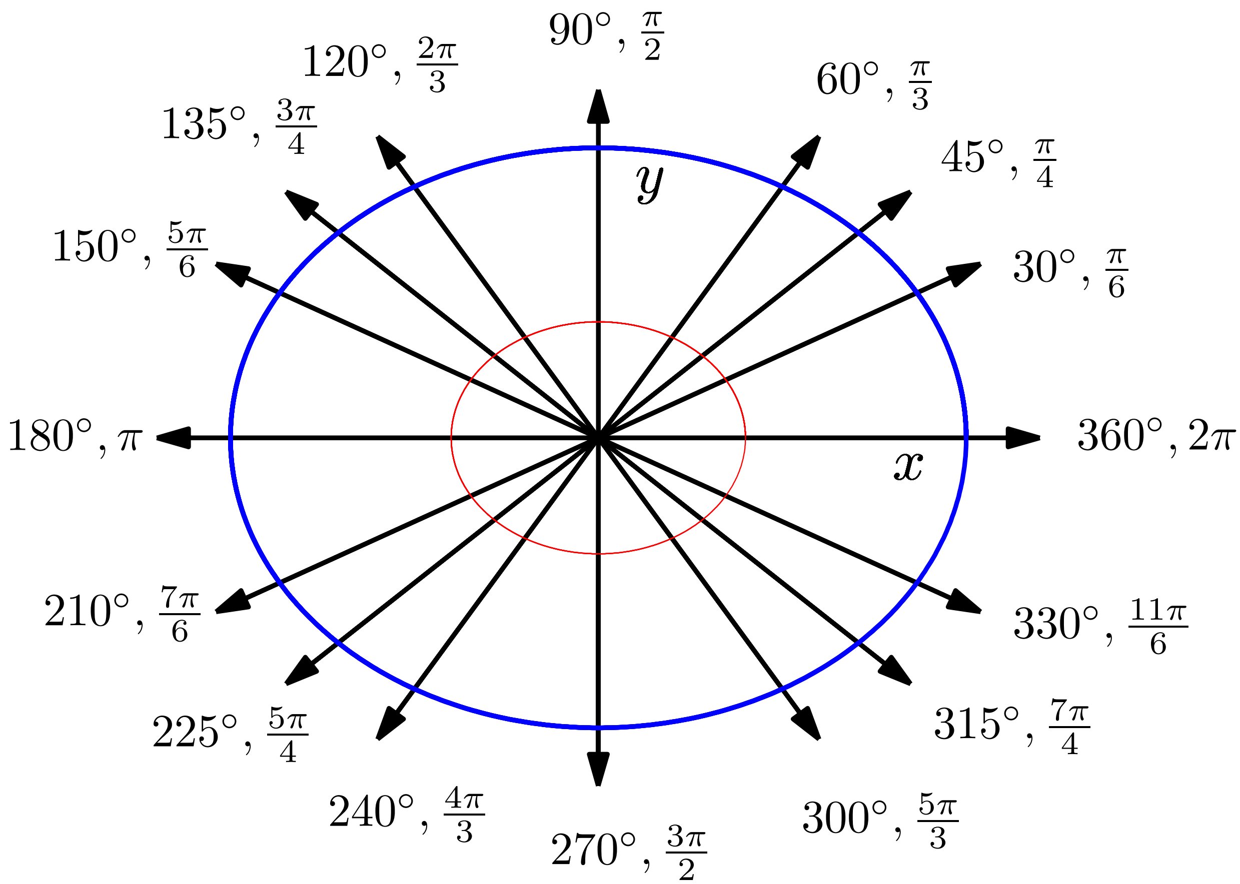
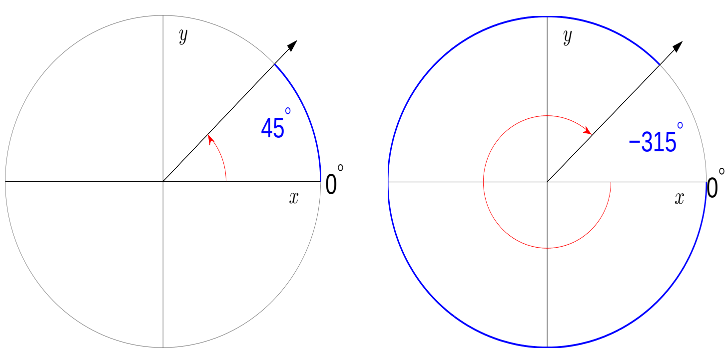


Figure 1: The common angles that live in the interval $(0, 360°]$

Definition 18. *Coterminal angles are angles in standard position that have the same initial side and same terminal side.*



Definition 19. *Angles α and β in standard position are coterminal if and only if there is an integer k such that $\alpha = \beta + k \cdot 360°$*

The measure of an angle in standard position is given. Find two positive angles and two negative angles that are coterminal with the given angle.

14. $\beta = 135°$ 14. ____________

15. $\beta = -\dfrac{\pi}{4}$ 15. ____________

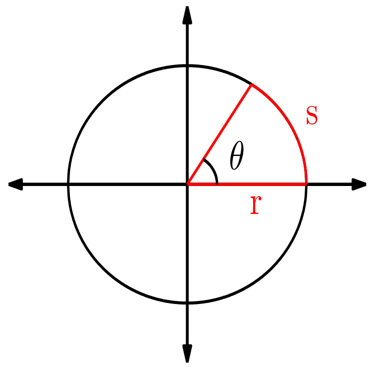
Find an angle in the open interval $[0°, 360°)$ **that is coterminal to the given angle.**

16. $\theta = 1110°$ 16. ____________

17. $\theta = -\dfrac{17\pi}{6}$ 17. ____________

Theorem 2 (Arc Length)**.** *The length s of an arc intercepted by a central angle of θ radians on a circle of radius r is given by*

$$s = r \cdot \theta$$



<u>CAUTION</u>: The theorem only holds when θ is measured in radians.

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18. How many revolutions will a car wheel of diameter 30 in. make as the car travels a distance of one mile?

19. A circular arc of length 18 feet subtends a central angle of 75 degrees. Find the radius of the circle in feet.

19. ____________

Theorem 3 (Area of a Circular Sector)**.** *In a circle of radius r, the area A of a sector with a central angle of θ radians is*

$$A = \frac{1}{2}r^2\theta$$

20. A central angle θ is subtended by an arc 10 cm long on a circle of radius 4 cm.

(a) Approximate the measure of θ in degrees.

(b) Find the area of the circular sector determined by θ.

Definition 20 (Linear Velocity and Angular Velocity)**.** *Suppose a point is in constant motion on a circle of radius r. Then its speed is*

$$v = \frac{s}{t} = \frac{arclength}{time}$$

where s *is the* arc length *determined by* $s = r\theta$.

An angle, denoted θ, is formed by the initial and terminal positions of the ray over some unit of time. The angular velocity of the point is

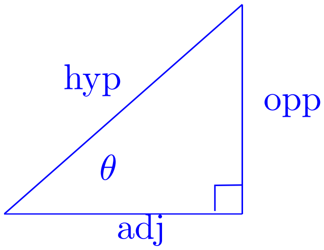
$$\omega = \frac{\theta}{t} = \frac{radians}{time}$$

Theorem 4 (Linear Velocity in Terms of Angular Velocity)**.** *If* v *is the linear velocity of a point on a circle of radius r, and ω (omega) is its angular velocity (in radians per time unit), then*

$$v = r\omega$$

21. What are the angular velocity in radians per second and the linear velocity in miles per hour of the tip of a 22-inch lawnmower blade that is rotating at 2500 revolutions per minute?

6.2 Right Triangle Trig **Name:** ____________________

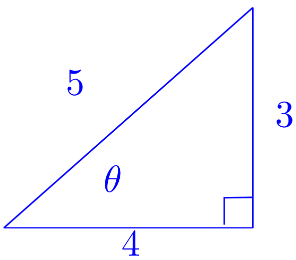


$$\sin(\theta) = \frac{opp}{hyp} \qquad \cos(\theta) = \frac{adj}{hyp} \qquad \tan(\theta) = \frac{opp}{adj}$$

$$\csc(\theta) = \frac{hyp}{opp} \qquad \sec(\theta) = \frac{hyp}{adj} \qquad \cot(\theta) = \frac{adj}{opp}$$

The six trigonometric functions of an acute angle theta (θ) are defined by ratios of side lengths of the right triangle containing theta. The six trigonometric functions of θ are the **sine, cosine, tangent, cotangent, secant,** and **cosecant** functions, abbreviated **sin, cos, tan, cot, sec** and **csc**, respectively.

1. Find the six trigonometric ratios of the angle θ.



2. If $\cos(\theta) = \frac{5}{6}$, sketch a right triangle with acute angle θ, and find the other five trigonometric ratios of θ.

Reciprocal Relations

$$\csc(\theta) = \frac{1}{\sin(\theta)} \qquad \sec(\theta) = \frac{1}{\cos(\theta)} \qquad \cot(\theta) = \frac{1}{\tan(\theta)}$$

Special Triangles

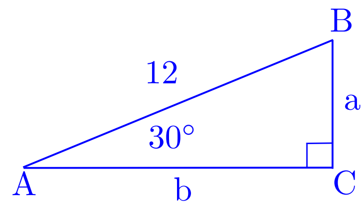
Use the special triangles to calculate the six trigonometric functions of θ for angles measuring 30°, 45° and 60°.

θ in degrees	θ in radians	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$	$\csc(\theta)$	$\sec(\theta)$	$\cot(\theta)$
30°							
45°							
60°							

To find the values of the trigonometric functions for angles that are not common angles, we use a calculator.

A triangle has six parts: 3 sides and 3 angles. To **solve a triangle** means to determine values for all six parts given some of the parts.

3. Solve triangle ABC.



4. A plane is flying within sight of the Gateway Arch in St. Louis, Missouri, at an elevation of 35,000 feet. The pilot would like to estimate her distance from the Gateway Arch. She finds that the angle of depression to a point on the ground below the arch is $22°$.

(a) What is the distance between a point on the ground and the arch?

(b) What is the distance between the plane and the base of the arch?

5. A man is lying on the beach, flying a kite. He holds the end of the kite string at ground level, and estimates the angle of elevation of the kite to be $50°$. If the string is 450 ft long, how high is the kite above the ground.

5.1 The Unit Circle **Name:** ________________________

Definition 21. *The **unit circle** is the circle of radius 1 centered about the origin in the xy-plane,*

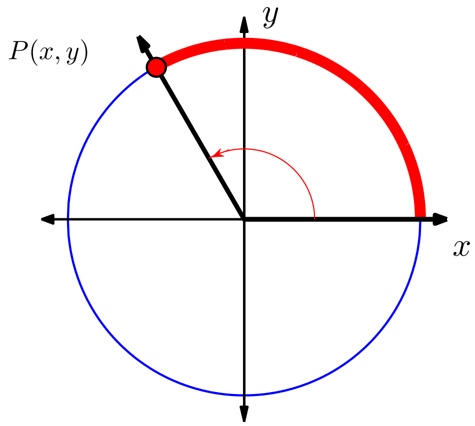
$$x^2 + y^2 = 1$$

1. Show that the point $P\left(-\frac{5}{13}, \frac{12}{13}\right)$ is a point on the unit circle.

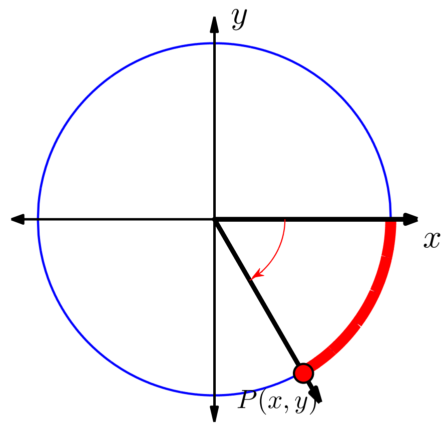
2. The point $P\left(\frac{\sqrt{3}}{2}, y\right)$ is a point on the unit circle in quadrant 4. Find its y-coordinate.

2. ____________

Definition 22. *Suppose θ is an angle in standard position. The point where the terminal side of the angle intersects the unit circle is called a* ***terminal point****, denoted $P(x, y)$.*



θ positive



θ negative

Find the terminal point on the unit circle determined by the given value of θ.

3. $\theta = -\dfrac{\pi}{4}$

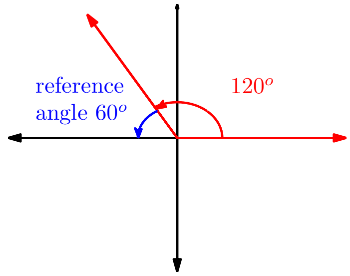
4. $\theta = -\pi$

5. $\theta = \dfrac{5\pi}{3}$

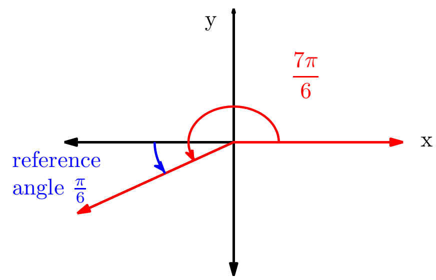
6. $\theta = \dfrac{3\pi}{2}$

Definition 23. *A quadrantal angle is an angle whose terminal side falls along an axis.*

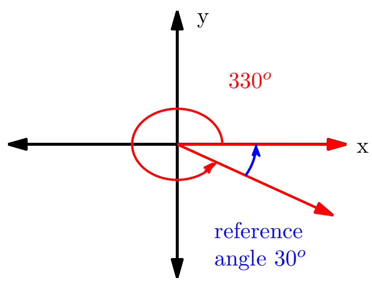
Definition 24. *Let θ be a non-quandrantal angle in standard position. The* ***reference angle*** *for θ, denoted θ' (called "theta prime"), is the acute angle that the terminal side of θ makes with the x-axis.*



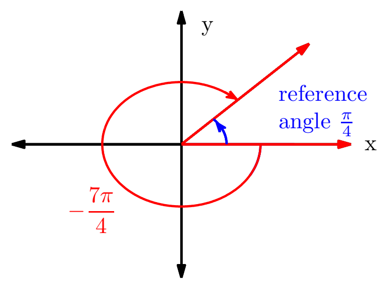
2nd quadrant angle



3nd quadrant angle



4nd quadrant angle



1st quadrant angle

How to find a Reference Angle, θ'

If the given angle, θ, is not in the interval $[0, 360°)$, then find the coterminal angle that is. Call that coterminal angle θ_{co}. Then if θ terminates in

- Q1, then $\theta' = \theta_{co}$
- Q2, then $\theta' = 180° - \theta_{co}$
- Q3, then $\theta' = \theta - 180°$
- Q4, then $\theta' = 360° - \theta_{co}$

If θ *is* in the interval $[0, 360°)$, then we use the above rule with $\theta_{co} = \theta$.

Find the reference angle and terminal point on the unit circle determined by the given value of θ.

7. $\theta = 315°$

8. $\theta = -240°$

9. $\theta = \dfrac{5\pi}{6}$

10. $\theta = 4$

11. Suppose that the terminal point determined by t is the point $\left(\frac{3}{4}, \frac{\sqrt{7}}{4}\right)$ on the unit circle. Find the terminal point determined by each of the following.

(a) $-t$

(b) $4\pi + t$

(c) $\pi - t$

(d) $t - \pi$

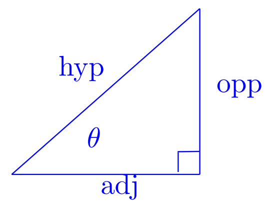
5.2 / 6.3 **Name:** ________________________

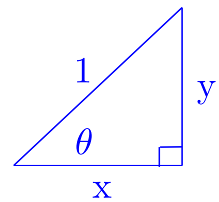
Recall the definitions of the sine, cosine, and tangent of an acute angle θ of a right triangle.

$$\sin(\theta) = \frac{opp}{hyp} \qquad \cos(\theta) = \frac{adj}{hyp} \qquad \tan(\theta) = \frac{opp}{adj}$$

We can use these definitions to establish the relationships that θ has with xy plane and the unit circle $x^2 + y^2 = 1$.

$$\cos(\theta) = x \qquad \sin(\theta) = y \qquad \tan(\theta) = \frac{y}{x}$$

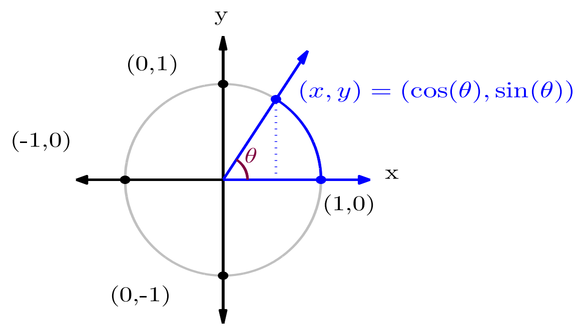




Theorem 5. *If θ is an angle in standard position and (x, y) is the point of intersection of the terminal side of θ with the unit circle, then*

$$x = \cos(\theta)$$
$$y = \sin(\theta)$$

Moreover,



$$\tan(\theta) = \frac{y}{x} \qquad \cot(\theta) = \frac{x}{y} \qquad \sec(\theta) = \frac{1}{x} \qquad \csc(\theta) = \frac{1}{y}$$

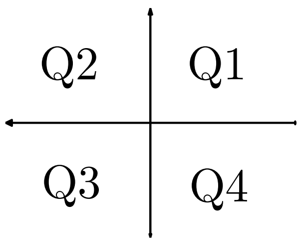
Find the six trigonometric functions for each.

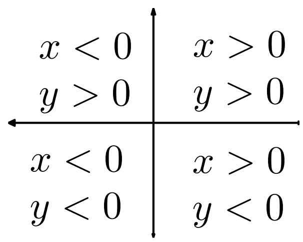
1. $\theta = \frac{\pi}{3}$

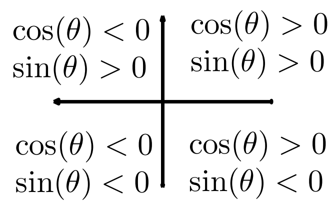
2. $\theta = \frac{\pi}{2}$

Find the six trigonometric functions of θ for the quadrantal angles listed in the table.

θ in degrees	θ in radians	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$	$\csc(\theta)$	$\sec(\theta)$	$\cot(\theta)$
$90°$							
$180°$							
$270°$							
$360°$							





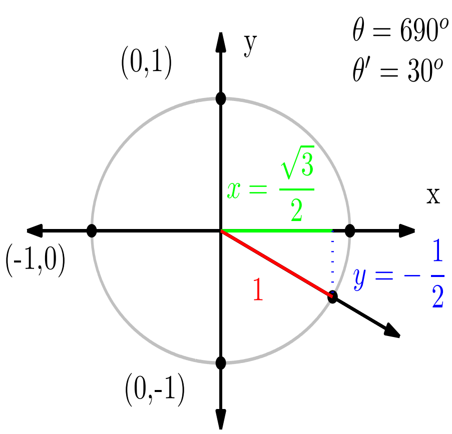


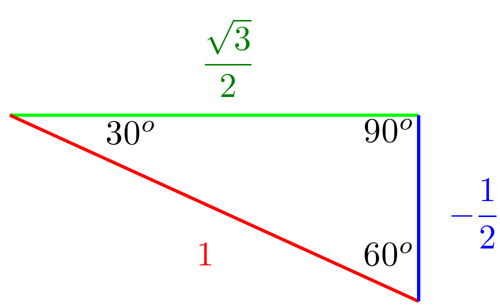
Evaluating Trigonometric Functions Using Ref. Angles

Let θ be an angle in standard position that is not a quadrantal angle. To find the values of the six trig functions for θ, we carry out the following steps.

1. Find the reference angle, θ' (called "theta prime"), associated with the given angle, θ.
2. Determine the sign ($+$ or $-$) of the trig function of θ by noting the quadrant in which the terminal side of θ lies.
3. The value of the trig function of θ is the same, except possibly for sign, as the value of the trig function of the reference angle, θ'.

Example. $\theta = 690^o$ has reference angle $\theta' = 30^o$ (and a 30^o-60^o-90^o ref triangle). So, $\sin(690^o) = y = -\frac{1}{2}$ since $y < 0$ in Q4. Also $\cos(690^o) = x = \frac{\sqrt{3}}{2}$ since $x > 0$ in Q4.





Evaluate without using a calculator.

3. $\sin 240°$

4. $\sin\left(\frac{5\pi}{6}\right)$

5. $\cos\left(\frac{5\pi}{6}\right)$

6. $\cos\left(\frac{7\pi}{3}\right)$

7. $\cot\left(\frac{7\pi}{3}\right)$

8. $\tan\left(-\frac{\pi}{3}\right)$

9. $\sec\left(\frac{11\pi}{3}\right)$

10. $\csc\left(\frac{7\pi}{4}\right)$

Evaluate without using a calculator.

11. $\sin 225°$

12. $\cos(-60°)$

13. $\sec 300°$

14. $\cot 210°$

15. $\sec 120°$

16. $\tan\left(-\frac{5\pi}{2}\right)$

17. $\sin\left(\frac{11\pi}{6}\right)$

18. $\cos\left(\frac{7\pi}{4}\right)$

The terminal point $P(x, y)$ determined by a real number t is given. Find sin(t), cos(t) and tan(t).

19. $\left(\frac{24}{25}, -\frac{7}{25}\right)$

Find an approximate value of the given trigonometric function by using a calculator.

20. $\tan\left(-\frac{5\pi}{12}\right)$ 20. ____________

21. sin 15 21. ____________

22. $\cos 15^o$ 22. ____________

23. $\sec 25^o$ 23. ____________

24. $\csc 25^o$ 24. ____________

Find the sign of the expression if the terminal point determined by t is in the given quadrant.

25. $\frac{\tan(t)\sin(t)}{\cot(t)}$ Quadrant 3 25. ____________

Even and Odd Properties

Sine, cosecant, tangent and cotangent are odd functions. Cosine and secant are even functions.

$$\sin(-t) = -\sin(t) \qquad \cos(-t) = \cos(t) \qquad \tan(-t) = -\tan(t)$$

$$\csc(-t) = -\csc(t) \qquad \sec(-t) = \sec(t) \qquad \cot(-t) = -\cot(t)$$

We will prove these identities later on once we have established the difference of angles formula.

We write $\sin^2(t)$ when we mean $\left(\sin(t)\right)^2$. In general, we write $\sin^n(t)$ instead of $\left(\sin(t)\right)^n$ whenever n is an integer not equal to -1. This convention applies to the other five trig functions.

Determine whether the function is even, odd, or neither.

29. $f(x) = x\sin^2(x)$, 29. ____________

30. $f(x) = x^3 \tan(x)$ 30. ____________

31. $f(x) = x^2 + \cos^2(x)$, 31. ____________

Reciprocal Identities

$$\csc(t) = \frac{1}{\sin(t)} \qquad \sec(t) = \frac{1}{\cos(t)} \qquad \tan(t) = \frac{\sin(t)}{\cos(t)} \qquad \cot(t) = \frac{1}{\tan(t)} \qquad \cot(t) = \frac{\cos(t)}{\sin(t)}$$

Pythagorean Identities

$$\sin^2(t) + \cos^2(t) = 1 \qquad \tan^2(t) + 1 = \sec^2(t) \qquad 1 + \cot^2(t) = \csc^2(t)$$

Derive the pythagorean identities from the definitions of sine and cosine.

32. Write $\tan(t)$ in terms of $\cos(t)$, where t is in Quadrant 3.

33. Express $\sin(\theta)$ in terms of $\cos(\theta)$.

34. Write $\tan(t)$ in terms of $\sin(t)$, where t is in Quadrant 2.

35. If $\tan(t) = \dfrac{2}{3}$ and t is in Quadrant 3, find $\cos(t)$.

Find the values of the trigonometric functions of *t* from the given information.

36. $\cos(t) = -\frac{4}{5}$, where t is in Quadrant 3

36. ____________

37. $\sec(t) = 2$, where $\sin(t) < 0$

37. ____________

38. $\tan(\theta) = -4$, where $\csc(\theta) < 0$

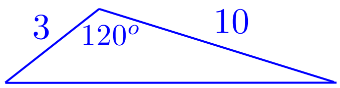
38. ____________

Theorem 6. *The area, A, of a triangle with sides of lengths a and b with included angle θ is*

$$A = \frac{1}{2}ab\sin(\theta)$$

proof:

39. Find the area of the triangle shown in the figure.



Graphing the Sine and Cosine Functions

Definition 25. *The generalized sine and cosine families of functions can be described by*

$$f(x) = A\ \sin[B\,(x - C)] + D \quad \text{and} \quad f(x) = A\ \cos[B\,(x - C)] + D$$

where A, B, C, and D are any real numbers.

- $|A|$ *represents the amplitude.*
- *The period is* $Pd = \dfrac{2\pi}{B}$
- *C is the phase shift, or the horizontal shift.*
- *D is the amount of vertical shift.*
- *QP is the quarter-point width given by* $QP = \dfrac{Pd}{4}$

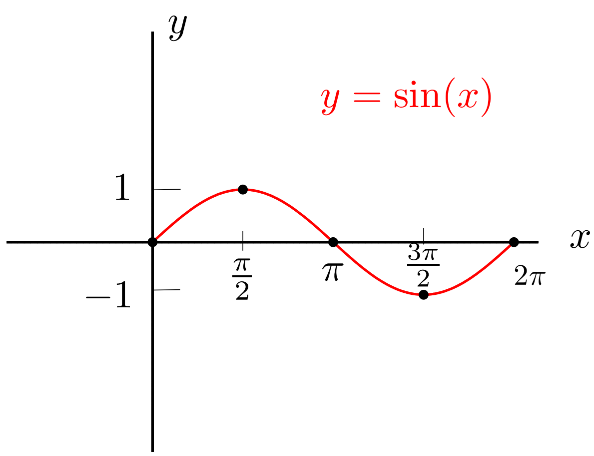
Graphing Algorithm (short version): One way to draw the graph of $y = A\ \sin(B\,[x - C]) + D$ or $y = A\ \cos(B\,[x - C]) + D$ is to carry out the following steps:

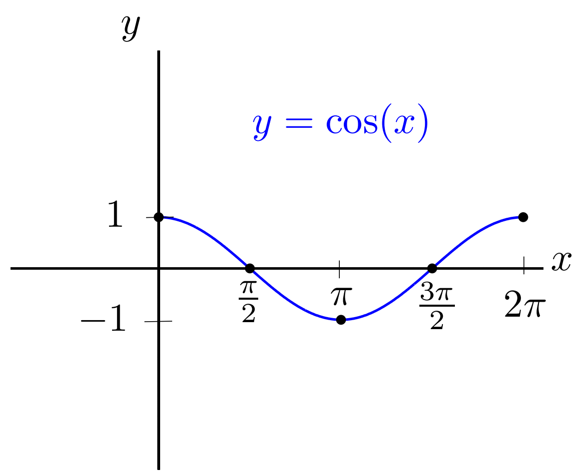
1. Graph one period of either $y = \sin(B\,x)$ or $y = \cos(B\,x)$ using $Pd = \frac{2\pi}{B}$.
2. Change the amplitude to follow $y = A\ \sin(B\,x)$ or $y = A\ \cos(B\,x)$. If $A < 0$, then a reflection (about the x-axis) must be applied to the graph.
3. Translate the cycle C units right if $C > 0$ when the argument of the sine (or cosine) function is factored in the form $[B(x - C)]$, or left if $C < 0$.
4. Shift the cycle D units up or down, depending upon if $D > 0$ or $D < 0$.
5. Locate the next four quarter-points using the quarter-point width (QP), then draw another cycle.

Graphing Algorithm (long version):

$$y = A\ \sin[B\,(x - C)] + D \quad \text{or} \quad y = A\ \cos[B\,(x - C)] + D$$

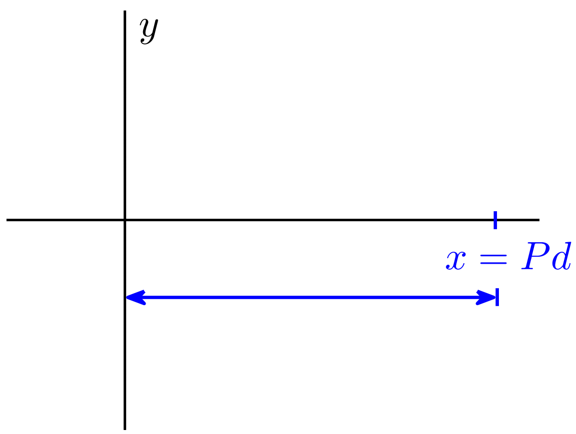
1. Recall the graphs of one fundamental cycle of $y = \sin(x)$ and $y = \cos(x)$, as well as the five key points associated with each of those graphs.

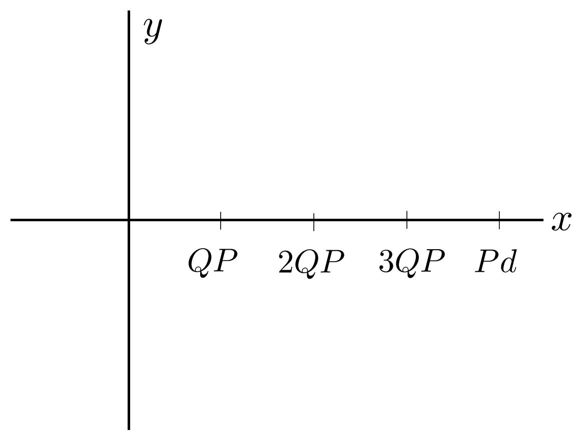




2. Identify and write down the values of A, B, C, and D corresponding to the given function. Then calculate the period (PD) and the quarter point width (QP) according to:

$$Pd = \frac{2\pi}{B} \qquad QP = \frac{Pd}{4}$$



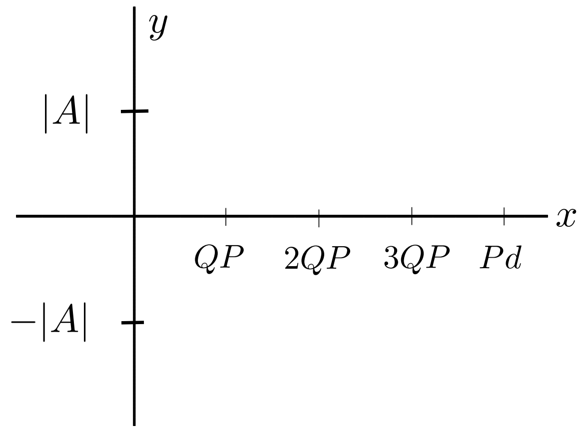


3. Locate and label $x = Pd$ along the x-axis.

4. Locate and label each of the quarter points (QP's)

5. Then locate and label $y = A$ and $y = -A$ along the y axis.

6. Identify which of the two graphs you were given: either sine or cosine. Recall the placement of the five key points on the graph of $y = \sin(x)$ or $y = \cos(x)$ for $x \in [0, 2\pi]$ (See step 1). Translate (mark) these five key points onto your graph, just as you would if we were to place the graph of $y = \sin(x)$ or $y = \cos(x)$ underneath your graph and draw a trace. The x-coordinate of the five key points will be located at $x = 0,\ QP,\ 2QP,\ 3QP,$ and PD; the y-coordinate of the five key points will be located at either $y = -A,\ 0,$ or A.



7. If A is negative, then apply a reflection of the five key points around the x-axis.

8. Horizontal Shift: now translate horizontally each of the five points on your graph C units. Recall that in order to determine the value of C we must factor the argument into the form $(B\,[x - C])$. Alternatively, we can just set the given argument of the sine or cosine function equal to zero and solve for x to find $x = C$. Then if $C > 0$ shift right, else if $C < 0$ shift left.

9. Vertical Shift: If D is not zero, vertically shift each of the five points on your graph D.

10. If you need to graph more than one period, repeat steps 3-6, starting step 3 by locating twice the length of Pd along the x-axis, followed by locating the three QP's between $x = Pd$ and $x = 2\,Pd$, using the quarter-point width (QP). You will then have your next five key points to draw the trace of one cycle of the sine or cosine curve through.

***Examples*: For each function given below, graph one cycle (period). Identify the domain and range of each function.**

1. $f(x) = 2\sin(x) - 1$

2. $f(x) = \sin(2x) + 2$

3. $f(x) = 2\sin(x - 1) + 1$

4. $f(x) = 2\sin\left(\frac{x}{2} - 1\right) + 1$

5. $f(x) = -3\cos\left(\frac{2}{3}x - \frac{\pi}{6}\right)$

5.4 more trig graphs **Name:** ________________________

Graph the given function.

1. $f(x) = \csc(x)$

2. What is the period, domain and range of $f(x)$?

3. Describe the behavior of the function around its vertical asymptotes.

Graph the given function.

4. $f(x) = \sec(x)$

5. What is the period, domain and range of $f(x)$?

6. Describe the behavior of the function around its vertical asymptotes.

Graph the given function and find the period, domain and range of f.

7. $f(x) = 2\csc\left(x - \dfrac{\pi}{3}\right) + 3$

Graph the given function.

8. $f(x) = \tan(x)$

9. What is the period, domain and range of $f(x)$?

10. Describe the behavior of the function around its vertical asymptotes.

Graph the given function and find the period, domain and range of f.

11. $f(x) = 3\tan\left(\frac{2}{3}x - \frac{\pi}{6}\right)$

Graph the given function.

12. $f(x) = \cot(x)$

13. What is the period, domain and range of $f(x)$?

14. Describe the behavior of the function around its vertical asymptotes.

5.5 Inverse Trig Functions **Name:** ______________________

Definition 26. *A function $f(x)$ with domain D and range R is a* <u>*one to one function*</u> *if either of the following equivalent conditions are satisfied.*

1. *Whenever $x_1 \neq x_2$ in D then $f(x_1) \neq f(x_2)$ in R.*
2. *Whenever $f(x_1) = f(x_2)$ in R, then $x_1 = x_2$ in D.*

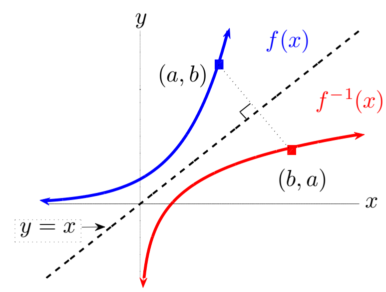
This means that a one to one function has the characteristic: for each functional value $f(x)$ in the range R there corresponds <u>EXACTLY ONE</u> element in the domain D.

Theorem 7. <u>*A function f is one to one*</u> *if and only if every horizontal line intersects the graph of f in at most one point.*

Properties of Inverse Functions

Suppose that f is a one to one function with domain D and range R. Then

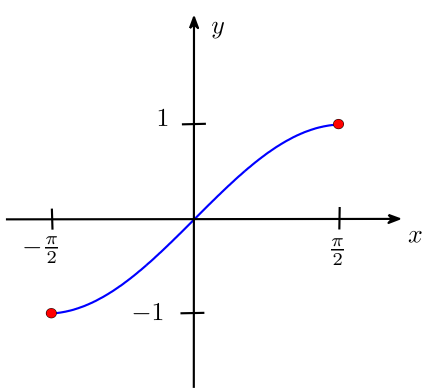
- The inverse function f^{-1} exists, and is unique.
- The domain of f^{-1} is the range of f.
- The range of f^{-1} is the domain of f.
- The statement $f(x) = y$ is equivalent to $f^{-1}(y) = x$
- If x is in the domain of $f(x)$, then $\left(f^{-1} \circ f\right)(x) = x$ and $\left(f \circ f^{-1}\right)(x) = x$



The Inverse Sine Function

The sine function is not one to one since it doesn't pass the horizontal line test. However, if we restrict the domain of the sine function to $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then $f(x) = \sin(x)$ becomes one to one so that a unique f^{-1} exists—written $f^{-1}(x) = \sin^{-1}(x)$.

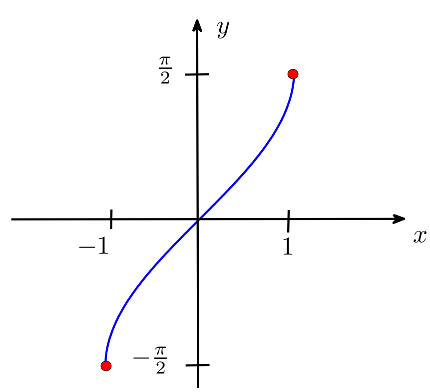
$f(x) = \sin(x)$



$\text{dom}\left(f\right)$ restricted to $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\text{range}\left(f\right) \equiv [-1, 1]$

$f^{-1}(x) = \sin^{-1}(x)$



$\text{dom}\left(f^{-1}\right) \equiv [-1, 1]$

$\text{range}\left(f^{-1}\right) \equiv \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

1. **Find each value.**

(a) $\sin^{-1}\left(\frac{1}{2}\right)$

(b) $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

(c) $\sin^{-1}\left(\frac{3}{2}\right)$

(d) $\sin^{-1}(-1)$

Properties of Inverse Sine

1. For $x \in [-1, 1]$, $\quad \sin\left(\sin^{-1}(x)\right) = x,$

2. For $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\quad \sin^{-1}\left(\sin(x)\right) = x,$

3. $\left[\ y = \sin^{-1}(x)\ \right] \Longleftrightarrow \left[\ \sin(y) = x\ \right]$

4. $\sin^{-1}(x) \neq \dfrac{1}{\sin(x)} = \left(\sin(x)\right)^{-1}$

5. The inverse sine of a value in $[-1, 1]$ will return a Q1 or Q4 (negative) angle or be $-\frac{\pi}{2}$ or $\frac{\pi}{2}$.

Use a calculator to find approximate to 3 decimal places.

2. $\sin^{-1}(-0.3)$ 2. ____________

3. $\sin^{-1}(-19)$ 3. ____________

Evaluate without a calculator.

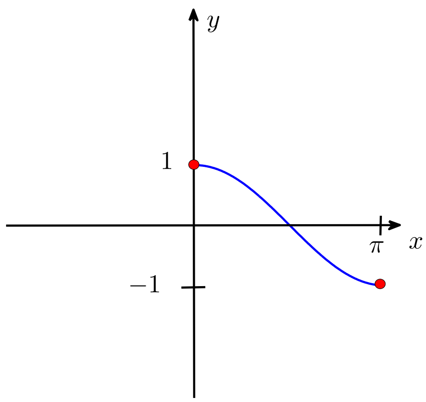
4. $\sin^{-1}\left(\sin\frac{\pi}{3}\right)$ 4. ____________

5. $\sin^{-1}\left(\sin\frac{3\pi}{4}\right)$ 5. ____________

The Inverse Cosine Function

The cosine function is not one to one since it doesn't pass the horizontal line test. However, if we restrict the domain of the cosine function to $x \in [0, \pi]$, then $f(x) = \cos(x)$ becomes one to one so that a unique f^{-1} exists—written $f^{-1}(x) = \cos^{-1}(x)$.

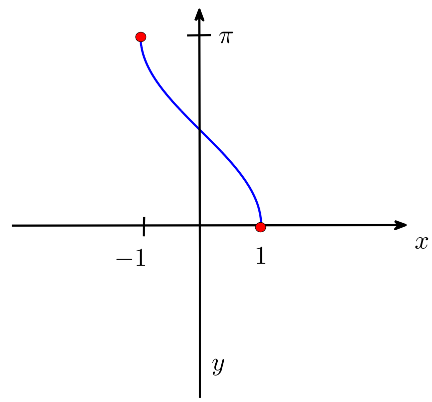
$f(x) = \cos(x)$



$\text{dom}\big(f\big)$ restricted to $x \in [0, \pi]$

$\text{range}\big(f\big) \equiv [-1, 1]$

$f^{-1}(x) = \cos^{-1}(x)$



$\text{dom}\big(f^{-1}\big) \equiv [-1, 1]$

$\text{range}\big(f^{-1}\big) \equiv [0, \pi]$

6. **Find each value.**

(a) $\cos^{-1}\left(\dfrac{1}{2}\right)$

(b) $\cos^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$

(c) $\cos^{-1}\left(\dfrac{3}{2}\right)$

(d) $\cos^{-1}(-1)$

Properties of Inverse Cosine

1. For $x \in [-1, 1]$, $\quad \cos\left(\cos^{-1}(x)\right) = x,$

2. For $x \in [0, \pi]$, $\quad \cos^{-1}\left(\cos(x)\right) = x,$

3. $\left[\, y = \cos^{-1}(x) \,\right] \Longleftrightarrow \left[\, \cos(y) = x \,\right]$

4. $\cos^{-1}(x) \neq \dfrac{1}{\cos(x)} = \left(\cos(x)\right)^{-1}$

5. The inverse cosine of a value in $[-1, 1]$ will return a Q1 or Q2 angle or be 0 or π.

Use a calculator to find approximate to 3 decimal places.

7. $\cos^{-1}(.8)$ — 7. ____________

8. $\cos^{-1}(-27)$ — 8. ____________

Evaluate without a calculator.

9. $\cos\left(\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$ — 9. ____________

10. $\cos^{-1}\left(\cos\frac{\pi}{6}\right)$ — 10. ____________

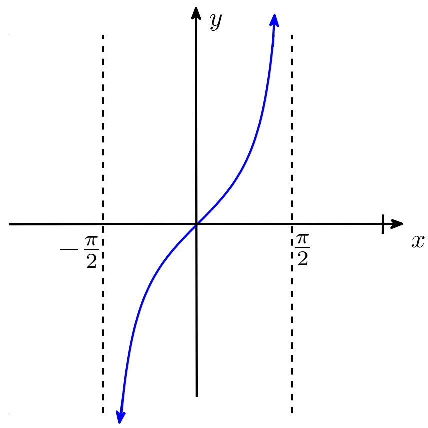
11. $\cos^{-1}\left(\cos\left(-\frac{\pi}{6}\right)\right)$ — 11. ____________

12. $\sin\left(\cos^{-1}\left(\frac{7}{25}\right)\right)$ — 12. ____________

The Inverse Tangent Function

The tangent function is not one to one since it doesn't pass the horizontal line test. However, if we restrict the domain of the tangent function to $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then $f(x) = \tan(x)$ becomes one to one so that a unique f^{-1} exists—written $f^{-1}(x) = \tan^{-1}(x)$.

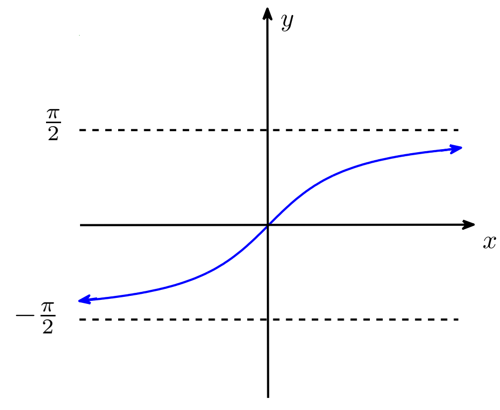
$f(x) = \tan(x)$



$\mathrm{dom}\left(f\right)$ restricted to $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\mathrm{range}\left(f\right) \equiv (-\infty, \infty)$

$f^{-1}(x) = \tan^{-1}(x)$



$\mathrm{dom}\left(f^{-1}\right) \equiv (-\infty, \infty)$

$\mathrm{range}\left(f^{-1}\right) \equiv \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

6. **Find each value.**

(a) $\tan^{-1}(0)$

(b) $\tan^{-1}\left(-\sqrt{3}\right)$

(c) $\tan^{-1}(-1)$

Properties of Inverse Tangent

1. For $x \in (-\infty, \infty)$, $\quad \tan\left(\tan^{-1}(x)\right) = x$,
2. For $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\quad \tan^{-1}\left(\tan(x)\right) = x$,
3. $\left[\, y = \tan^{-1}(x) \,\right] \Longleftrightarrow \left[\, \tan(y) = x \,\right]$
4. $\tan^{-1}(x) \neq \dfrac{1}{\tan(x)} = \left(\tan(x)\right)^{-1}$
5. The inverse tangent of a value in $(-\infty, \infty)$ will return a Q1 or Q4 (negative) angle.

Find the exact value of the expression (without a calculator), if it is defined.

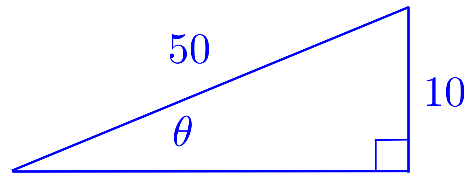
13. $\tan\left(\cos^{-1}\left(\frac{12}{13}\right)\right)$

14. $\tan\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

15. $\sin\left(\tan^{-1}\left(\frac{3}{\sqrt{3}}\right)\right)$

6.4 **Name:** ______________

1. Find the angle θ in the triangle shown in the figure.



2. A 40-ft ladder leans against a building. If the base of the ladder is 6 ft from the base of the building, what is the angle formed by the ladder and the building?

3. A 96-ft tree casts a shadow that is 120 ft long. What is the angle of elevation of the sun?

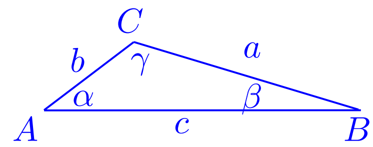
4. Find all angles θ between 0° and 180° satisfying the given condition.

   (a) $\sin(\theta) = \dfrac{1}{4}$ **(b)** $\cos(\theta) = \dfrac{1}{8}$

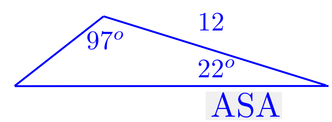
5. Write $\sin\left(\cos^{-1}(x)\right)$ and $\tan\left(\cos^{-1}(x)\right)$ as algebraic expressions in x for $-1 \leq x \leq 1$.

6.5 Law of Sines **Name:** ______________________

In a previous section, we used the definitions of sine, cosine and tangent to solve *right* triangles. Now we extend our study by solving triangles that are not right triangles—called *oblique* triangles. Recall that there are six parts to any triangle: three side lengths and three angles. For instance, consider ΔABC (right) which has the three sides a, b and c and angles α, β and γ (the first three letters of the greek alphabet). In this section, we will be given three parts of a triangle and asked to find the remaining three parts. Here is a list of possible scenarios:

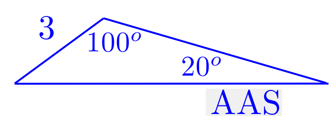


ASA Two angles and a side length between them is given. AAS or SAA are considered the same as ASA, be–cause if we know any two angles, then we know all three angles.



AAA Three angle measures are given.

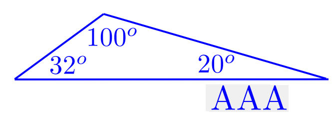
SSS Three side lengths are given.



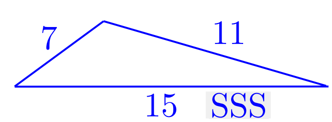
SAS Two sides and an angle between the two sides is given.

SSA Two sides and an angle opposite one of the sides is given.

An example of each scenario is given (right). We will examine each case individually. We have two different formulas we use in Sections 6.5 & 6.6: the **Law of Sines** and the **Law of Cosines**. The table below summarizes which procedure we use for each case.



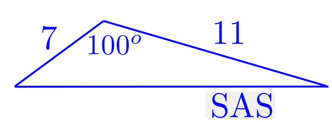
ASA — Use **Law of Sines**.



SAA — Use **Law of Sines**.

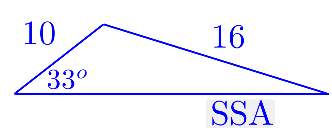
AAA — No unique solution.

SSS — **Law of Cosines** if the triangle inequality is satisfied.
— No solution if the triangle inequality is <u>not</u> satisfied.



SAS — Use **Law of Cosines**.

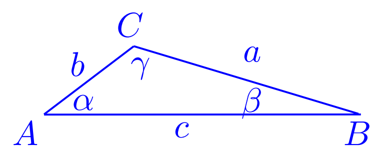
SSA — Use **Law of Cosines**: there are 0, 1 or 2 triangles.



For any case giving at least one angle, the triangle has no solution if the given angle is greater than or equal to 180^o.

Law of Sines
For any ΔABC,

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$$



***proof*:**

Sketch the triangle with the given conditions, then solve it.

1. $\alpha = 23^o$, $\beta = 110^o$, $c = 50$

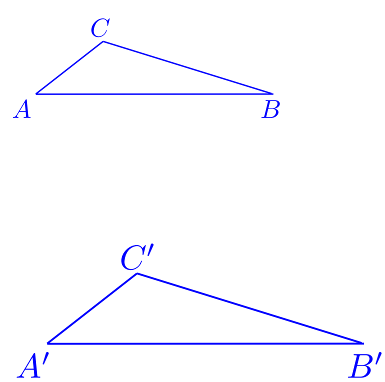
Sketch the triangle with the given conditions, then solve it.

2. $a = 20$, $\alpha = 32^o$, $\beta = 121^o$

3. Points A and B on opposite sides of the river are endpoints for a proposed bridge. To find the length of this proposed bridge, a point C is located 100ft from point A. It is then determined that $\alpha = 58^o$, $\gamma = 49^0$. Find the length of the footbridge.

AAA

We have seen that the two cases ASA and AAS are easily solved by the Law of Sines. Another possible case arises when three angles are given. However, from what we know about similar triangles, they have the same shape but are not necessarily the same size. This means their corresponding angles have equal measure, so we conclude that the triangle cannot be solved without knowing the length of at least one side.



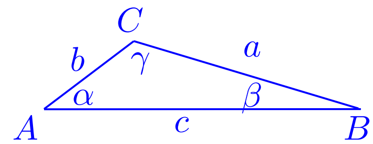
6.6 Law of Cosines **Name:** ______________________

Law of Cosines
For any ΔABC,

$$a^2 = b^2 + c^2 - 2bc \cdot \cos(\alpha)$$

$$b^2 = a^2 + c^2 - 2ac \cdot \cos(\beta)$$

$$c^2 = a^2 + b^2 - 2ab \cdot \cos(\gamma)$$



***proof*:**

Sketch the triangle with the given conditions, then solve it.

1. $a = 25,\ b = 18,\ c = 21$

Sketch the triangle with the given conditions, then solve it.

2. $a = 52.0$, $b = 28.3$, $\gamma = 28.5^o$

3. The Geezer Bandit has just committed the crime of the century and left 30 minutes ago traveling 460 mi/h in his Learjet with a heading of 120^o (measured clockwise from north). At this instant, Batman is located 150 mi from the crime scene on a bearing of 140^o. How far apart are the two planes?

SSA

SSA is a special case because it is possible to obtain one of three conclusions:

1. No triangle with the given measurements and properties exists.
2. There is a unique triangle satisfying the given measurements.
3. There are two triangles with the given measurements.

SSA is called the "ambiguous case" because it is not clear at the outset of solving the triangle which one of these cases is given.

SSA

If you use the Law of Cosines on the ambiguous case, the resulting equation will be a quadratic, and **the number of positive solutions to that quadratic dictates how many triangles there are with the given properties.**

0 positive solutions $\Longrightarrow$ no such triangle exists.
1 positive solution $\Longrightarrow$ the triangle is unique.
2 distinct positive solutions $\Longrightarrow$ there are two different triangles.

Sketch the triangle with the given conditions, then solve it.

4. $a = 3.0$, $b = 2.0$, $\alpha = 110^o$

5. $a = 1.5$, $b = 2.0$, $\alpha = 40^o$

6. $a = 3.0$, $b = 1.5$, $\beta = 40^o$

7. $a = 2.0$, $b = 3.0$, $\beta = 40^o$

8. An enemy aircraft target is 100 miles N40^{o}E of your unit's outpost, traveling due west at 240 mi/h. How long will it take (to the nearest minute) before the plane is 90 miles from your fixed position (within range of your weapons systems)?

7.1 Trig Identities **Name:** ________________________

In this section, we simplify trigonometric expressions and prove trigonometric identities. A trigonometric identity is an equation that is true for all angles in the domain of the equation. We will be given a bunch of trigonometric identities we've never seen before and asked to prove that each follows from the fundamental identities summarized in the box below.

Reciprocal Identities

$$\csc(t) = \frac{1}{\sin(t)} \qquad \sec(t) = \frac{1}{\cos(t)} \qquad \tan(t) = \frac{\sin(t)}{\cos(t)} \qquad \cot(t) = \frac{1}{\tan(t)} \qquad \cot(t) = \frac{\cos(t)}{\sin(t)}$$

Pythagorean Identities

$$\sin^2(t) + \cos^2(t) = 1 \qquad \tan^2(t) + 1 = \sec^2(t) \qquad 1 + \cot^2(t) = \csc^2(t)$$

Even and Odd Properties

$$\sin(-t) = -\sin(t) \qquad \cos(-t) = \cos(t) \qquad \tan(-t) = -\tan(t)$$

$$\csc(-t) = -\csc(t) \qquad \sec(-t) = \sec(t) \qquad \cot(-t) = -\cot(t)$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - t\right) = \cos(t) \qquad \tan\left(\frac{\pi}{2} - t\right) = \cot(t) \qquad \sec\left(\frac{\pi}{2} - t\right) = \csc(t)$$

$$\cos\left(\frac{\pi}{2} - t\right) = \sin(t) \qquad \cot\left(\frac{\pi}{2} - t\right) = \tan(t) \qquad \csc\left(\frac{\pi}{2} - t\right) = \sec(t)$$

1. Draw a right triangle and locate an acute angle t. Explain how you can obtain all six cofunction identities from this triangle for $0 < t < \frac{\pi}{2}$.

Simplify each expression.

2. $\dfrac{\sec(x)}{\csc(x)}$

2. ______________

3. $\cos^2\theta\,(1 + \tan^2\theta)$

3. ______________

4. $\dfrac{\cot\theta}{\csc\theta - \sin\theta}$

4. ______________

5. $\dfrac{\sin\theta}{\csc\theta} + \dfrac{\cos\theta}{\sec\theta}$

5. ______________

Simplify each expression.

6. $\dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{1 + \sin\theta}$

6. ____________

7. $\cos^3\theta + \sin^2\theta\cos\theta$

7. ____________

Verify each identity.

8. $\cos\theta\,(\sec\theta - \cos\theta) = \sin^2\theta$

9. $1 + \sin^2\theta = 2 - \cos^2\theta$

Verify each identity.

10. $\dfrac{1}{1-\sin x}-\dfrac{1}{1+\sin x}=2\sec x\tan x$

11. $\dfrac{1-\cos(-\theta)}{1+\cos(-\theta)}=\left(\dfrac{1-\cos\theta}{\sin\theta}\right)^2$

12. $\dfrac{1-\tan(-\theta)}{1+\tan(-\theta)}=\dfrac{\sec^2\theta+2\tan\theta}{2-\sec^2\theta}$

Verify each identity.

13. $\dfrac{1+\cos(-\theta)}{\cos\theta} = \dfrac{\tan^2\theta}{\sec\theta - 1}$

14. $\dfrac{\sin^2\theta - \cos^2\theta}{\sin\theta + \cos\theta} = \sin\theta - \cos\theta$

15. $\dfrac{\sin^3\theta - \cos^3\theta}{\sin\theta - \cos\theta} = 1 + \sin\theta\cos\theta$

16. Make the substitution $x = 2\tan\theta$ into $\dfrac{1}{x^2\sqrt{4+x^2}}$ and simplify.

7.2 Sum & Difference Formulas

Name: ______________________

Sum & Difference of Angles Identities

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$

Find the exact value of each without a calculator.

1. $\cos 75^o$

2. $\tan 345^o$

3. $\sin \dfrac{11\pi}{12}$

4. Prove all of the Sum & Difference of Angles Identities

5. Verify the identity $\dfrac{1 + \tan x}{1 - \tan x} = \tan\left(\dfrac{\pi}{4} + x\right)$

6. Write $\sin\left(\cos^{-1}(x) + \tan^{-1}(y) \right)$ as an algebraic expression in x and y. Assume $-1 \leq x \leq 1$ and y is any real number.

7. Evaluate $\cos\left[\sin^{-1}\left(\dfrac{4}{5}\right) + \cos^{-1}\left(\dfrac{3}{5}\right) \right]$ without a calculator.

8. Evaluate $\sin(\theta+\phi)$, where $\sin\theta = \dfrac{12}{13}$ with θ in Quadrant 2 and $\tan\phi = \dfrac{3}{4}$ with ϕ in Quadrant 3.

9. Express $\sqrt{3}\sin x - 3\cos x$ in the form $k\sin(x+\phi)$

7.3 Double & Half Angle Identities **Name:** ________________________

Sum & Difference of Angles Identities

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$= 1 - 2\sin^2\theta$$

$$= 2\cos^2\theta - 1$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

1. ***proof***

2. Write $\cos 3\theta$ in terms of $\cos\theta$.

3. If $\cos x = -\frac{2}{3}$ and x is in Quadrant 2, find $\cos(2x)$ and $\sin(2x)$.

4. Prove $\frac{\sin 3x}{\sin x \cos x} = 4\cos x - \sec x$

Power Reducing Formulas

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$$

***proof*:**

5. Express $\cos^2 x \sin^2 x$ in terms of the first power of cosine.

Half Angle Identities

$$\sin\frac{1}{2}\theta = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\frac{1}{2}\theta = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\frac{1}{2}\theta = \frac{\sin\theta}{1+\cos\theta}$$

proof

6. Find the exact value of $\cos\left(\frac{\pi}{8}\right)$

7. Find the exact value of $\sin\left(\frac{5\pi}{12}\right)$

8. Find $\tan\frac{1}{2}\theta$ if $\sin\theta = \frac{2}{5}$ and θ is in Quadrant 2.

9. Write $\tan\left(2\cos^{-1}(x)\right)$ as an algebraic expression in x.

10. Evaluate $\sin 2\theta$, where $\cos\theta = -\frac{2}{5}$ with θ in Quadrant 2.

7.4—7.5 More Trig Eqns **Name:** ________________________

Our work with trigonometric equations thus far has been limited to proving identities. Identities are true statements for all angles, θ in the domain of each equation. In this section, we will use our experience to find solutions to conditional trigonometric equations. Conditional equations are equations which are true statements for certain values of θ.

For each equation, determine
(a) all solutions in the interval $[0, 2\pi)$**, and**
(b) all solutions.

1. $\sin\theta = \frac{1}{2}$

2. $\cos\theta = -\frac{\sqrt{2}}{2}$

3. $\sin\theta = 0.7$

4. $\tan\theta = 2$

5. $2\sin\theta - 1 = 0$

6. $\tan^2\theta - 3 = 0$

7. $2\cos^2\theta - 7\cos\theta + 3 = 0$

8. $5\sin\theta\cos\theta + 4\cos\theta = 0$

9. $1 + \sin\theta = 2\cos^2\theta$

10. $\sin 2\theta - \cos\theta = 0$

11. $\cos\theta + 1 = \sin\theta$

12. $\cos\theta = \sin\theta$

13. $2\sin 3\theta - 1 = 0$

14. $\sqrt{3}\tan\frac{\theta}{2} - 1 = 0$

Polar Forms of Complex Numbers
De Moivre's Theorem **Name:** ________________________

Graph the complex number, $z = a+bi$, and find its modulus, $r = \left|z\right| = \sqrt{a^2 + b^2}$.

1. $z = -3i$

2. $z = 7$

3. $z = 7 - 3i$

4. $z = -1 - \dfrac{\sqrt{3}}{3}i$

Sketch the set in the complex plane.

5. $\psi \equiv \left\{ z = a + bi \;\middle|\; a \leq -1, \;\; b \geq 2 \right\}$

6. $\Upsilon \equiv \left\{ z = a + bi \;\middle|\; |z| < 3 \right\}$

7. $\Phi \equiv \left\{ z = a + bi \;\middle|\; a + b < 2 \right\}$

Write the complex number in polar form with argument θ between 0 and 2π.

8. $z = 4 + 4i$

9. $z = 4\sqrt{3} - 4i$

10. $z = -3 - 4i$

11. $z = 2 - 5i$

Find the product $z_1 \cdot z_2$ and the quotient z_1/z_2.

12. $z_1 = 3\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right), \qquad z_2 = 3\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right)$

13. $z_1 = \sqrt{2}\left(\cos\left(75°\right) + i\sin\left(75°\right)\right), \qquad z_2 = 3\sqrt{2}\left(\cos\left(60°\right) + i\sin\left(60°\right)\right)$

Find the indicated power using De Moivre's Theorem.

14. $\left(1 - \sqrt{3}i\right)^5$

15. $\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^{15}$

16. $\left(2\sqrt{3} + 2i\right)^{-5}$

17. $\left(1 - 2i\right)^{1/3}$

18. $\sqrt[3]{i}$

Find the complex solutions of each equation.

19. $z^3 + 8 = 0$

20. $x^4 + 16 = 0$

Plane Curves and Parametric Equations

A pair of parametric equations is given.

- **Sketch the curve represented by the parametric equations.**
- **Find a rectangular coordinate equation for the curve by eliminating the parameter variable, *t*.**

21. $x(t) = 6t - 4, \quad y(t) = 3t, \quad \text{where } t \geq 0$

22. $x(t) = \cos(t), \quad y(t) = \sin(t),$

23. $x(t) = t^2, \quad y(t) = t^4 + 1,$

24. $x(t) = \cos(t), \quad y(t) = \cos(2t),$

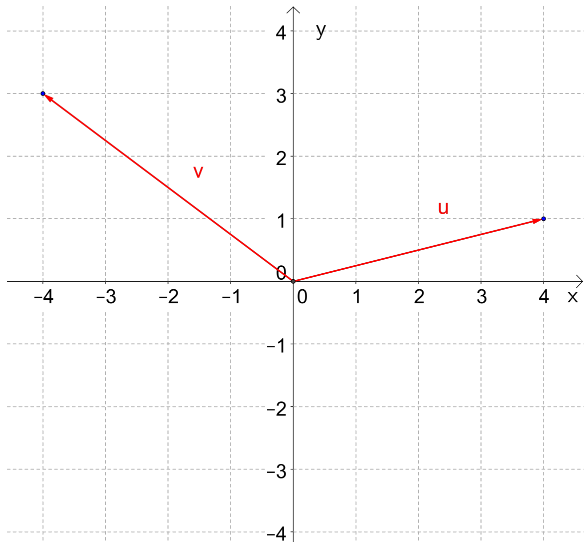
Vectors in Two Dimensions **Name:** ______________________

Sketch the vector indicated.

1. $2\vec{u}$

2. $\vec{v} - 2\vec{u}$

3. $\vec{u} - \vec{v}$



Express the vector with initial point P and terminal point Q in component form.

4. $P(1,1)$ and $Q(3,3)$

5. $P(-3,-4)$ and $Q(-2,3)$

Sketch the given vector with initial point at $(2,1)$.

6. $\vec{u} = \langle -1, 2 \rangle$

7. $\vec{v} = \langle 4, -5 \rangle$

Write the given vector in terms of i and j.

8. $\vec{v} = \langle -3, 7 \rangle$

9. $\vec{u} = \langle 8, -5 \rangle$

Find $\left|\vec{u}\right|$, $\left|\vec{v}\right|$, $\left|2\vec{u}\right|$, $\left|\vec{u}+\vec{v}\right|$, $\left|\vec{u}-\vec{v}\right|$ **and** $\left|\vec{u}\right| - \left|\vec{v}\right|$

10. $\vec{u} = 2\mathbf{i} - 1\mathbf{j}$ and $\vec{v} = 3\mathbf{i} - 2\mathbf{i}$

Find the horizontal and vertical components of the vector with given length and direction, and write the vectors in terms of i and j

11. $\left|\vec{u}\right| = 50$ and $\theta = 120°$

Find the magnitude and direction of the vector.

12. $\vec{u} = \langle -12, 5 \rangle$

13. $\vec{u} = 3\mathbf{i} - 7\mathbf{j}$

14. A man pushes a lawn mower with a force of 30 lb. exerted at an angle of 30° to the ground. Find the horizontal and vertical components of the force.

15. A jet is flying in a direction N 20° E with a speed of 500 mi/h. Find the north and east components of velocity.

16. A boat heads in the direction N 72° E. The speed of the boat relative to the water is 24 mph. The water is flowing directly south. It is observed that the true direction of the boat is directly east.

    (a) Express the velocity of the boat relative to the water as a vector in component form.

    (b) Find the speed of the water and the true speed of the boat.

Find the dot product, $\vec{u} \cdot \vec{v}$ between the two vectors $\vec{u}$ and $\vec{v}$. Then find the angle between $\vec{u}$ and $\vec{v}$.

17. $\vec{u} = \langle 2, 6 \rangle$ and $\vec{v} = \langle -7, 1 \rangle$

18. $\vec{u} = -3\mathbf{i}$ and $\vec{v} = 2\mathbf{i} + 2\mathbf{j}$

19. $\vec{u} = \langle 2, 3 \rangle$ and $\vec{v} = \langle 4, 5 \rangle$

Determine whether the vectors are orthogonal.

20. $\vec{u} = \langle 0, -5\sqrt{3} \rangle$ and $\vec{v} = \langle 4, 0 \rangle$

21. $\vec{u} = 2\mathbf{i}$ and $\vec{v} = -7\mathbf{j}$

22. The force $\vec{F} = 20\mathbf{i} + 13\mathbf{j}$ moves an object 6 ft. along the x-axis in the positive direction. Find the work done if the unit of force is the pound.

23. A man pulls a wagon horizontally by exerting a force of 35 lb. on the handle. If the handle makes an angle of 60° with that horizontal, find the work done in moving the wagon 100 ft.