Prerequisite Review 1

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June 19, 2013

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Mathematical Theorems

Definition

A theorem is a statement that has been proven on the basis of previously established statements.

Basic Set Definitions

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Example 1

{ 1,2,3,4} denotes a set containing the four numbers 1, 2, 3 and 4.

Basic Set Definitions

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Then $A \cup B$ is equal to what set?

$A \cup B =$

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$A \cup B = \{1,2,3,4,5,6\}$

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Example 3 Suppose $A = \{1,2,3,4\}$ and $B = \{2,4,20\}$. Then $A \cap B$ is equal to what set?

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$A \cap B = \{2,4\}$

Number Types

Definition

The natural numbers,

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$$

consists of the counting numbers, where the ellipsis $(\ldots)$ indicates that the set goes on to infinity, or that there is no upper bound (largest number) in the set.

Number Types

Definition

The set of whole numbers,

$$\mathbb{W} = \{0, 1, 2, 3, 4, \ldots\}$$

is the set of natural numbers unioned with zero, written $\mathbb{W} = \mathbb{N} \cup \{0\}$.

Number Types

Definition

The set of integers,

$$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

is also known as all the positive and negative whole numbers.

Number Types

Definition

A rational number is any number that can be written as a fraction where both the numerator and denominator are integers (and the denominator is not zero). The set of rational numbers is written symbolically as

$$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; \text{a and b are any integers, and } b \neq 0 \right\}$$

Note that any integer "a" is a rational number since $a = \frac{a}{1}$.

Number Types

HUGE note: When a rational number (fraction) is represented as a decimal number, then

1. it has a finite number of digits to the right of the decimal point; for example, $\frac{5}{4} = 1.25$, OR

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1. it has a finite number of digits to the right of the decimal point; for example, $\frac{5}{4} = 1.25$, OR
2. it has an infinite number of digits to the right of the decimal point *AND* and those digits have a repeating pattern, for example $\frac{1}{3} = 0.\overline{3}$ and $\frac{2}{37} = 0.054054054\cdots = 0.\overline{054}$.

Number Types

Definition

Real numbers that are not rational, for example, $\sqrt{2}$, $\sqrt[3]{3}$, and π are called irrational numbers. The set of irrational numbers is denoted $\mathbb{I}$.

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Real numbers that are not rational, for example, $\sqrt{2}$, $\sqrt[3]{3}$, and π are called irrational numbers. The set of irrational numbers is denoted $\mathbb{I}$.

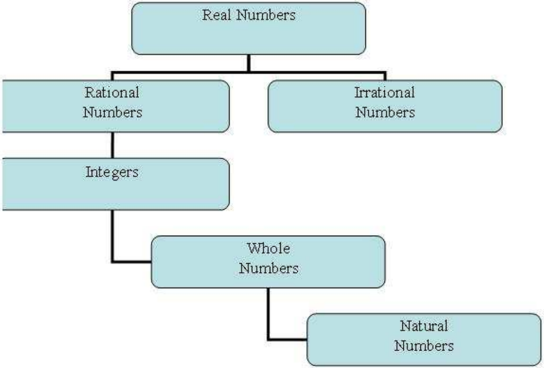
HUGE note: When an irrational number is represented as a decimal number, then

① it has an infinite number of digits to the right of the decimal point,
② and those digits do not have a repeating pattern.

Number Types

Definition

The set of real numbers, denoted $\mathbb{R}$, is the set $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$, that is the set of rationals unioned with the irrationals. Each real number can be uniquely represented as a decimal, and we associate each real number with a distinct point on a coordinate (number) line.



Real Numbers

Rational Numbers

Irrational Numbers

Integers

Whole Numbers

Natural Numbers

Opposites

Theorem

Every real number has an opposite. The sum of a real number and its opposite is zero. Opposites are often called additive inverses.

Example The opposite of 5 is -5, and $5 + (-5) = 0$

Reciprocals

Theorem

Every non-zero real number has a reciprocal. The product of a real number and its reciprocal is one. Reciprocals are often called multiplicative inverses.

Example The multiplicative inverse of 5 is $\frac{1}{5}$, and $5 \cdot \frac{1}{5} = 1$

Absolute value

Definition (Absolute Value)

The absolute value of a real number is its distance from 0 on the number line. If $|x|$ represents the absolute value of x, then

$$| x | = \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{if } x < 0. \end{cases}$$

The absolute value of a real number is never negative.

Example $| 5 | = 5$ and $| -5 | = 5$.

Theorem (Properties of Absolute value)

For any real numbers a and b:

1. $|a| \geq 0$
2. $|-a| = |a|$
3. $|a \cdot b| = |a| \cdot |b|$
4. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ *provided* $b \neq 0$

Addition

To add two real numbers with *the same sign:*

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- Add the absolute values and use the common sign.

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Examples

$7 + 5 = 12$

$-7 + (-5) = -12$

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Examples

$$7 + (-5) = 2$$

$$-7 + 5 = -2$$

Subtraction

We use addition to define subtraction.

Definition

Suppose a and b represent any two real numbers. Then

$$a - b = a + (-b)$$

To subtract b, add the opposite of b.

Multiplication

To multiply two real numbers:

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$-7 \cdot 5 = -35$

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Division

We use multiplication to define division.

Definition

Suppose a and b represent any two real numbers, and that $b \neq 0$. Then

$$\frac{a}{b} = a \cdot \frac{1}{b}$$

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Suppose a and b represent any two real numbers, and that $b \neq 0$. Then

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Example

$$\frac{3}{4} = 3 \cdot \frac{1}{4}$$

Properties of Real Numbers

Closure

Suppose a and b represent any real numbers, then $a + b$ and $a \cdot b$ are real numbers too.

	For Addition	For Multiplication
Commutative	$a + b = b + a$	$a \cdot b = b \cdot a$
Associative	$a + (b + c) = (a + b) + c$	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$
Identity	$0 + a = a$	$1 \cdot a = a$
Inverse	$a + (-a) = 0$	$a \cdot \left(\frac{1}{a}\right) = 1$
Mult. Prop of Zero	$0 \cdot a = 0$	

The Distributive Property

Theorem (Distributive Property of Multiplication)

Multiplication distributes over addition. For example,

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

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The distributive property says that multiplication distributes over addition. For example, notice that $3 \cdot (2+5)$ simplifies to the same number as $3 \cdot 2 + 3 \cdot 5$.

$$3 \cdot \underbrace{(2+5)}_{\quad} = 3(7) = 21$$

and

$$3 \cdot 2 + 3 \cdot 5 = 6 + 15 = 21$$

The Distributive Property

The distributive property says that multiplication distributes over addition. For example, notice that $3\cdot(2+5)$ simplifies to the same number as $3\cdot 2+3\cdot 5$.

$$3\cdot\underbrace{(2+5)} = 3(7) = 21$$

and

$$3\cdot 2+3\cdot 5 = 6+15 = 21$$

Therefore,

$$3(2+5) = 3\cdot 2+3\cdot 5$$

Notice in the expression $3\cdot(2+5)$ that each number inside the parenthesis is multiplied by 3.

The Distributive Property

Example Use the Distributive Property on the algebraic expression, $3 \cdot (x - 1)$. Assume x represents a real number.

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Solution:

$$3 \cdot (x - 1) = 3 \cdot \Big(x + (-1)\Big)$$ Definition of Subtraction

$$= 3 \cdot x + 3 \cdot (-1)$$ Distributive Property

$$= 3x + (-3)$$

$$= 3x - 3$$ Definition of Subtraction

Theorem

Suppose a and b are any real numbers. Then

$$-\frac{a}{b} = \frac{-a}{b} \text{ and } -\frac{a}{b} = \frac{a}{-b}$$

Exponential Notation

Recall the following terminology related to multiplication.

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$$\underset{\text{factor}}{\underset{\uparrow}{4}} \times \underset{\text{factor}}{\underset{\uparrow}{6}} = \underset{\text{product}}{\underset{\uparrow}{24}}$$

The numbers we are multiplying are called **factors**.

Exponential Notation

Recall the following terminology related to multiplication.

$$\underset{\substack{\uparrow \\ \text{factor}}}{4} \times \underset{\substack{\uparrow \\ \text{factor}}}{6} = \underset{\substack{\uparrow \\ \text{product}}}{24}$$

The result of multiplying the factors is called the **product**.

Exponential Notation

In the product $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$, notice that 2 is a factor several times.

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$\underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{\text{2 is a factor 5 times}}$ can be written as 2^5. (Read as "two to the fifth power.")

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$\underbrace{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{\text{2 is a factor 5 times}}$ can be written as 2^5. (Read as "two to the fifth power.")

base

This is called **exponential notation**. The **exponent**, 5, indicates how many times the **base**, 2, is a factor.

Exponential Expressions

Helpful Hint

An exponent applies only to its base. For example, $4 \cdot 2^3$ means $4 \cdot 2 \cdot 2 \cdot 2$

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Helpful Hint

Dont forget that 2^4, for example is not $2 \cdot 4$. The expression 2^4 means repeated multiplication of the same factor.

$$2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16 \text{ whereas } 2 \cdot 4 = 8$$

The Order of Operations

Example: Simplify $6 + 2 \cdot 30$. Do you multiply or add first?

The Order of Operations

When evaluating a mathematical expression, we will perform the operations in the following order:

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When evaluating a mathematical expression, we will perform the operations in the following order:

1. Begin with the expression in the innermost parenthesis or brackets and work our way out.
2. Change exponential expressions into repeated multiplication.
3. Multiply or divide in order from left to right.
4. Add or subtract in order from left to right.

The Order of Operations

When evaluating or simplifying an algebraic or arithmetic expression, we always use the ORDER OF OPERATIONS. In other words we evaluate the expression in the following order:

$$\begin{aligned} P &- \mathit{Parenthesis} \\ E &- \mathit{Exponents} \\ M &- \mathit{Multiplication} \\ D &- \mathit{Division} \\ A &- \mathit{Addition} \\ S &- \mathit{Subtraction} \end{aligned}$$

The operations $+, -, \times, \div$ must be performed from left to right! The acronym

PEMDAS helps us to recall the ORDER OF OPERATIONS.

Example

Example Recall that for any real number a, it is always true that $-a = (-1) \cdot a$. Use the Order of Operations and $-a = (-1) \cdot a$ to simplify the arithmetic expression $5 - 10^2$.

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Solution:

$$5 - 10^2 = 5 + (-10^2) \qquad \text{Definition of Subtraction}$$

$$= 5 + \left((-1) \cdot 10^2\right) \qquad \text{Since } -a = (-1) \cdot a$$

$$= 5 + \left((-1) \cdot 10 \cdot 10\right) \qquad \text{Since } 10^2 = 10 \cdot 10$$

$$= 5 + (-100) = -95$$

Theorem (Properties of Equality)

For any real numbers a and b:

1. $a = a$	*Reflexive property*
2. If $a = b$, then $b = a$	*Symmetric property*
3. If $a = b$ and $b = c$, then $a = c$	*Transitive property*
4. If $a = b$, then a and b may be be substituted for one another in any expression involving a and b	*Substitution property*

Rational Exponents

Definition

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For instance, the square of 4 is 16 because 4^2 or $4 \cdot 4 = 16$. The square of -4 is also 16 because $(-4)^2 = (-4) \cdot (-4) = 16$.

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Definition

The reverse process of squaring is **finding a square root**.

For example, a square root of 16 is 4 because $4^2 = 16$. A square root of 16 is also -4 because $(-4)^2 = (-4) \cdot (-4) = 16$.

Theorem

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Definition

We use the symbol $\sqrt{}$, called a **radical sign**, to indicate the positive (or "principal") square root.

For example,

$$\sqrt{25} = 5 \text{ because } 5^2 = 25 \text{ and 5 is positive.}$$

$$\sqrt{9} = 3 \text{ because } 3^2 = 9 \text{ and 3 is positive.}$$

Theorem

Every positive number has two square roots.

For instance, the square roots of 25 are 5 and -5.

Note: it is a common mistake to assume that an expression like $\sqrt{25}$ indicates both square roots, 5 and -5. The expression $\sqrt{25}$ indicates only the positive square root of 25, which is 5. If we want the negative square root, we must use a negative sign in front of the radical sign.

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We write the negative square root of 25 as $-\sqrt{25}$ (which is –5).

Definition (Square Root of a Number)

The **square root,** $\sqrt{}$, of a positive number a is the positive number b whose square is a. In symbols,

$$\sqrt{a} = b \quad \text{if} \quad b^2 = a$$

For example,

$$\sqrt{36} = 6 \quad \text{if} \quad 6^2 = 36$$

Find the square root of each.

$\sqrt{100}$

$\sqrt{64}$

$-\sqrt{81}$

$-\sqrt{121}$

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Find the square root of the following.

$\sqrt{-16}$

$\sqrt{-16}$ is not a real number since there is no real number we can raise to the second power and obtain -16.

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For example,

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Find the square root of each.

$$\sqrt{\frac{1}{4}}$$

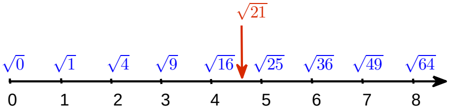
$$-\sqrt{\frac{49}{16}}$$

$$-\sqrt{\frac{4}{25}}$$

Definition

Numbers like $\frac{1}{4}$, $\frac{4}{25}$, 9 and 36 are called **perfect squares** because their square root is a whole number or a fraction.

A square root such as $\sqrt{21}$ cannot be written as a whole number or a fraction since 21 is not a perfect square. It can be approximated by estimating, by using a table, or by using a calculator. We can however, estimate what two whole numbers $\sqrt{21}$ is between.



Finding Functional Values

For each function, find the functional value, $f(10)$.

1. $f(x) = \sqrt{4x - 5}$
2. $f(z) = \sqrt{15 - z}$

Simplifying $\sqrt{a^2}$

Theorem

For any real number a,

$$\sqrt{a^2} = |a|$$

Simplifying $\sqrt{a^2}$

Theorem

For any real number *a*,

$$\sqrt{a^2} = |a|$$

Simplify each radical expression as much as possible.

$\sqrt{4x^2}$

$\sqrt{(t+4)^2}$

$\sqrt{x^2 - 4x + 4}$

$\sqrt{a^{10}}$

Definition (Cube Root of a Number)

The **cube root**, $\sqrt[3]{}$, of a number a is the number b whose cube is a. In symbols,

$$\sqrt[3]{a} = b \quad \text{if} \quad b^3 = a$$

For example,

$$\sqrt[3]{27} = 3 \quad \text{since} \quad 3^3 = 27$$

Find the cube root of each.

$\sqrt[3]{8}$

$\sqrt[3]{-8}$

$-\sqrt[3]{\frac{1}{8}}$

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For example,

$$\sqrt[3]{27} = 3 \quad \text{since} \quad 3^3 = 27$$

Definition

An expression like $-\sqrt[3]{\frac{1}{8}}$ involving a radical sign is called a **radical expression**. In the radical expression $-\sqrt[3]{\frac{1}{8}}$, the number 3 is called the **index** of the radical, and $\frac{1}{8}$ is called the **radicand**.

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For example,

$$\sqrt[3]{27} = 3 \quad \text{since} \quad 3^3 = 27$$

Definition (More Fine Print)

The index of a radical must be a positive integer greater than 1. If no index is written, it is assumed to be 2.

Definition (Fourth Root of a Number)

The **fourth root**, $\sqrt[4]{}$, of a positive number a is the number b such that

$$\sqrt[4]{a} = b \quad \text{if} \quad b^4 = a$$

For example,

$$\sqrt[4]{16} = 2 \quad \text{since} \quad 2^4 = 16$$

Find the fourth root of each.

$\sqrt[4]{1}$

$-\sqrt[4]{\frac{1}{16}}$

$\sqrt[4]{-16}$

Definition (Fourth Root of a Number)

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Find the fourth root of each.

$\sqrt[4]{1}$

$-\sqrt[4]{\frac{1}{16}}$

$\sqrt[4]{-16}$

$\sqrt[4]{-16}$ is not a real number since there is no real number we can raise to the fourth power and obtain -16.

There are also fifth roots, sixth roots, seventh roots, and so on. As a generalization, we call $\sqrt[n]{a}$ the n^{th} root of a.

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The n^{th} **root**, $\sqrt[n]{a}$, of a positive number a is the number b such that

$$\sqrt[n]{a} = b \quad \text{if} \quad b^n = a$$

We can use the following chart to help summarize the fine print of the definition.

n	a	$\sqrt[n]{a}$	$\sqrt[n]{a^n}$
Even	Positive	Positive	a
	Negative	Not a real number	$-a$
Odd	Positive	Positive	a
	Negative	Negative	a

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Even	Positive	Positive	a
	Negative	Not a real number	$-a$
Odd	Positive	Positive	a
	Negative	Negative	a

It needs to be clear that we cannot take an even root of a negative number!!!

Definition

A radical function is a function that can be described by a radical expression.

Finding the Domain:

- When the index of the the expression is odd, the domain is the set of real numbers, $\mathbb{R}$
- When the index of the the expression is even, the domain is the set of values that make the radicand non-negative upon substitution.

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Finding the Domain:

- When the index of the the expression is odd, the domain is the set of real numbers, $\mathbb{R}$
- When the index of the the expression is even, the domain is the set of values that make the radicand non-negative upon substitution.

Find the domains for each function.

- $f(x) = \sqrt{x-1}$
- $f(x) = \sqrt[4]{1-2x}$
- $f(x) = \sqrt[3]{x-1}$

Rational Numbers as Exponents

We have encountered exponential expressions like 2^3 and $(-2x)^5$ which have integers exponents. But what about expressions like $2^{1/2}$ and $(3x)^{3/5}$ which have integers exponents?

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For example, we can rewrite $\sqrt{16} = 16^{1/2}$ and $\sqrt{x} = x^{1/2}$.

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Theorem (The Fractional Exponent Rule)

$$\sqrt[d]{a^n} = a^{n/d}$$

Write each expression as a radical expression and then simplify the result, if possible.

$(-8)^{1/3}$

$-(144)^{1/2}$

$(-144)^{1/2}$

$\left(-144^{\frac{1}{2}} \right)$

$-(81)^{1/4}$

$(xyz)^{1/4}$

$(25x^{16})^{1/2}$

Theorem (The Fractional Exponent Rule)

$$\sqrt[d]{a^n} = a^{n/d}$$

Write an equivalent expression using exponential notation.

$\sqrt[5]{5ab}$

$\sqrt[2]{2x}$

$\sqrt[7]{\dfrac{x^3 y}{4}}$

Theorem (Positive Rational Exponent Rule)

$$a^{n/d} \quad \textit{means} \quad \left(\sqrt[d]{a}\right)^n, \quad \textit{or} \quad \sqrt[d]{a^n}$$

Simplify as much as possible.

$9^{3/2}$

$16^{3/4}$

$8^{-2/3}$

$\left(\frac{16}{81}\right)^{-3/4}$

Theorem (Positive Rational Exponent Rule)

$$a^{-n/d} \quad \textit{means} \quad \frac{1}{a^{n/d}}$$

Write an equivalent expression with positive exponents and simplify as much as possible.

$(5xy)^{-4/5}$

$4x^{-2/3}y^{1/5}$

$\left(\frac{3x}{7y}\right)^{-5/2}$

Use laws of exponents to simplify as much as possible.

$$x^{\frac{1}{3}} \cdot x^{\frac{5}{3}}$$

$$y^{-3/8} \cdot y^{5/12} \cdot y^{7/9}$$

$$(x^{2/3})^{3/4}$$

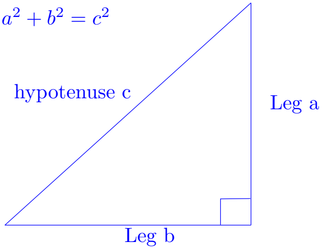
$$\frac{x^{3/4}}{x^{2/3}}$$

$$\frac{(x^{1/3}y^{-3})^6}{x^4y^{10}}$$

The Pythagorean Theorem and Square Roots

Theorem

If a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse, then $a^2 + b^2 = c^2$.



The Pythagorean Equation, $c^2 = a^2 + b^2$, can be written as $c = \sqrt{a^2 + b^2}$

Product Property for Radicals

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$$

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$$\begin{aligned}\sqrt{50} &= \sqrt{25 \cdot 2} \\ &= \sqrt{25} \cdot \sqrt{2} \qquad \text{by the product property}\end{aligned}$$

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If that was true, then

$$\sqrt{16} = \sqrt{4+4+4+4} = \sqrt{4} + \sqrt{4} + \sqrt{4} + \sqrt{4} = 2+2+2+2 = 8$$

Quotient Property for Radicals

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For example,

$$\sqrt{\frac{2}{25}} = \frac{\sqrt{2}}{\sqrt{25}}$$

$$= \frac{\sqrt{2}}{5}$$ by the quotient property

When we write radical expressions in simplified form it means we are writing the expressions so that they are easiest to work with.

Simplified Form for Radical Expressions

A radical expression is in simplified form if

1. None of the factors of the radicand can be written as powers greater than or equal to the index—that is, no perfect squares can be factors of the quantity under a square root sign, no perfect cubes can be factors of what is under a cube root sign, and so forth;
2. There are no fractions under the radical sign; and
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2. There are no fractions under the radical sign; and
3. There are no radicals in the denominator.

Try these on your own! Write each radical expression in simplified form. Assume that any variables represent positive quantities.

- $\sqrt{50x^2y^3}$
- $\sqrt[3]{32a^4b^6}$
- $\sqrt[3]{54x^5y^8}$
- $\sqrt{75m^5n^8}$
- $\sqrt[4]{80x^3y^8}$

Rationalizing the Denominator

Try these on your own! Write each radical expression in simplified form. Assume that any variables represent positive quantities.

$$\sqrt{\frac{50}{9x^2}}$$

$$\sqrt{\frac{5}{6}}$$

$$\sqrt{\frac{4}{5}}$$

$$-\sqrt{\frac{5}{2}}$$

Rationalizing the Denominator

Try these on your own! Write each radical expression in simplified form. Assume that any variables represent positive quantities.

$$\frac{2\sqrt{3x}}{5y}$$

$$\frac{3\sqrt{5x}}{2y}$$

$$\sqrt{\frac{12x^5y^3}{5z}}$$

$$\frac{7}{\sqrt[3]{4}}$$

$$\frac{5}{\sqrt[3]{9}}$$

$$\sqrt{\frac{48x^3y^4}{7z}}$$

Addition or Subtraction of Radical Expressions

We have been adding and subtracting polynomials by combining like terms. We do the same with radical expressions.

Addition or Subtraction of Radical Expressions

Definition

Two radical terms are said to be similar, or like terms if they have the same index *and* same radicand.

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Identify whether or not each radical expression contains like terms or not.

$$2\sqrt[3]{5}+\sqrt[3]{5}$$

$$\sqrt{3}+5\sqrt{3}-2\sqrt{3}$$

$$2\sqrt[7]{4}+\sqrt[7]{9}$$

Adding or Subtracting Radical Expressions

To add or subtract radical expressions, put each in simplified form and apply the distributive property, if possible. We can add only like radicals. We must write each expression in simplified form for radicals before we can tell if the radicals are similar.

Try these on your own! Write each radical expression in simplified form. Then combine like radicals. Assume that any variables represent positive quantities.

$$\sqrt{7} - 3\sqrt{7}$$

$$6x\sqrt{a} + 5x\sqrt{a}$$

$$7\sqrt[6]{7} - \sqrt[6]{7} + 4\sqrt[6]{7}$$

$$5x\sqrt{8} + 3\sqrt{32x^2} - 5\sqrt{50x^2}$$

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$$2\sqrt[3]{x^8y^6} - 3y^2\sqrt[3]{8x^8}$$

$$5a^2\sqrt{27ab^3} - 6b\sqrt{12a^5b}$$

$$b\sqrt[3]{24a^5b} + 3a\sqrt[3]{81a^2b^4}$$

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- A **constant** is either a fixed number, such as 5, or a letter or symbol that represents a fixed number.

- An **algebraic expression** is any combination of variables, constants, grouping symbols, exponents and arithmetic operators.

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- To **evaluate an algebraic expression**, substitute a numerical value for each variable into the expression and simplify the result by applying the order of operations in a left to right fashion.

Definition

A **term** is either a single number or variable, or the product or quotient of several numbers or variables separated from another term by a plus or minus sign in an overall expression.

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A **constant** is a single number, such as 8 or 9.

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$$ax^n,$$

where a is a constant that is any real number, x is a variable, and n is a whole number $(0, 1, 2, \ldots)$.

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For instance,

$$3, \qquad 5x, \qquad 7x^4, \qquad \text{and} \qquad 9x^{200}$$

are all examples of monomial functions.

Definition

We call n the **degree** of the monomial.

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The degree of a nonzero constant is zero. Because $0 = 0x = 0x^2 = 0x^3 = \ldots$, we cannot assign a degree to the 0. Therefore, we say 0 has no degree.

Monomial	Coefficient	Degree
3	3	0
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- $p(x) = 4x^{-3}$ is not a monomial because the exponent of the variable, x, is -3 and -3 is not a whole number.
- $p(x) = 2x^{1/3}$ is not a monomial because the exponent of the variable is $1/3$, and $1/3$ is not a whole number.

A polynomial of degree n is an expression of the form:

$$a_n\, x^n + a_{n-1}\, x^{n-1} + \cdots + a_3\, x^3 + a_2\, x^2 + a_1\, x + a_0$$

where n is a non-negative integer and $a_n \neq 0$.

- The numbers $a_n, a_{n-1}, \ldots, a_3, a_2, a_1, a_0$ are the *COEFFICIENTS* of the polynomial.
- a_0 is called the *CONSTANT TERM*.
- $a_n\, x^n$ is called the *LEADING TERM* of the polynomial.
- a_n is called the *LEADING COEFFICIENT* of the polynomial.
- n is called the *DEGREE* of the polynomial.

Definition

A **polynomial** is a monomial or a sum of monomials.

11	monomial
$3x^4$	monomial
$2x^2 + 1$	binomial
$5x^3 + x - 1$	trinomial
$x^{1/2} + 5$	is not a polynomial
$\sqrt[5]{x+5}$	is not a polynomial
$\dfrac{1}{x-1}$	is not a polynomial

Definition

The **degree of polynomial expression** is the degree of the leading term (the term which has x raised to the largest power).

Polynomial Function	Degree	Leading Term	Leading Coefficient	Constant term
$-2x^4 - 3x - 5$	4	$-2x^4$	-2	-5
$x^5 - 3x^6 - 10x - 4$	6	$-3x^6$	-3	-4
$5x^{10} - 8x^3 - 10x + 5$	10	$5x^{10}$	5	5
$17x + 4$	1	$17x$	17	4
24	0	24	24	24

Polynomial addition and subtraction is as one would expect.

Example Compute the difference $(x^2 - 5x) - (3x^2 - 4x - 1)$

$$
\begin{aligned}
&(x^2 - 5x) - (3x^2 - 4x - 1) = \\
&= (x^2 - 5x) - 1(3x^2 - 4x - 1) && \text{since } -a = (-1)\cdot a \\
&= (x^2 - 5x) + (-1)(3x^2 - 4x - 1) && \text{since } a - b = a + (-b) \\
&= (x^2 - 5x) - 3x^2 + 4x + 1 && \text{distr. prop} \\
&= \ x^2 - 5x \ - 3x^2 + 4x + 1 && \text{assoc. prop} \\
&= (x^2 - 3x^2) + (-5x + 4x) + 1 && \text{comm. and assoc. props} \\
&= \boxed{-2x^2 - x + 1} && \text{addn closure prop}
\end{aligned}
$$

Definition (Like Terms)

Like terms are terms that contain the same variable(s) raised to the same power(s). Like terms can be combined or collected together.

Example Identify the like terms in $4x^3 + 5x - 7x^2 + 2x^3 + x^2$

Solution:

like terms: $4x^3$ and $2x^3$ same variable and exponent

like terms: $-7x^2$ and x^2 same variable and exponent

Example Identify the like terms in $8x^2y^2 + 4x - 6x^5 + 2x^2y^2$

Solution:

like terms: $8x^2y^2$ and $2x^2y^2$ same variables and exponents

Multiplying Polynomials

The product of two binomials results in four terms before the like terms are combined. The acronym "Foil" stands for *FIRST, OUTER, INNER, LAST*, and should remind you how to compute the product of two binomials. Consider the following product:

$$(a+b)(c+d) = a(c+d)+b(c+d) = \overbrace{ac}^{\text{F}} + \overbrace{ad}^{\text{O}} + \overbrace{bc}^{\text{I}} + \overbrace{bd}^{\text{L}}$$

The product of the two binomials consists of four terms:

- the product of the *FIRST* term of each (ac),
- the product of the *OUTER* term of each (ad),
- the product of the *INNER* term of each (bc), and
- the product of the *LAST* term of each (bd).

2 Examples:

a.) $(x+4)\cdot(2x-3)$ b.) $(3\sqrt{6}-2\sqrt{5})^2$

Example: Multiply $(x^2 - 3x + 4) \cdot (2x - 3)$

Solution

$$\begin{aligned}
&(x^2 - 3x + 4) \cdot (2x - 3) = \\
&= (2x - 3) \cdot (x^2 - 3x + 4) && \text{comm prop } \times \\
&= 2x \cdot (x^2 - 3x + 4) + (-3) \cdot (x^2 - 3x + 4) && \text{distr. prop } \times \\
&= 2x^3 - 6x^2 + 8x - 3x^2 + 9x - 12 && \text{distr. prop } \times \\
&= 2x^3 + (-6x^2 - 3x^2) + (8x + 9x) - 12 && \text{comm., assoc. } + \\
&= \boxed{2x^3 - 9x^2 + 17x - 12} && \text{addn closure prop}
\end{aligned}$$

Example: Simplify $\dfrac{3}{5-\sqrt{2}}$ so that no radicals are in the denominator.

Solution

$$
\begin{aligned}
\frac{3}{5-\sqrt{2}} &= \frac{3}{5-\sqrt{2}}\cdot(1) \\
&= \frac{3}{5-\sqrt{2}}\cdot\frac{5+\sqrt{2}}{5+\sqrt{2}} && \text{since } \frac{5+\sqrt{2}}{5+\sqrt{2}}=1 \\
&= \frac{3(5+\sqrt{2})}{5^2-(\sqrt{2})^2} && \text{since } \boxed{(a-b)\cdot(a+b)=a^2-b^2} \\
&= \frac{15+3\sqrt{2}}{25-2} && \text{Distr. prop } \& \ \sqrt[n]{a^n}=a \\
&= \frac{15+3\sqrt{2}}{23} && \text{closure} \\
&= \boxed{\frac{15}{23}+\frac{3\sqrt{2}}{23}} && \text{since } \frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}
\end{aligned}
$$

#1 RULE: FACTOR OUT THE GCF

Factoring reverses multiplication. Consider the polynomial expression $6x^2 - 3x$, whose two terms have a greatest common factor, $3x$.

$$\begin{aligned} 6x^2 - 3x &= ({\color{red}3x}) \cdot (2x) - ({\color{red}3x}) \cdot (1) \\ &= {\color{red}3x} \cdot (2x - 1) \qquad \text{since } {\color{red}a} \cdot {\color{blue}b} - {\color{red}a} \cdot {\color{blue}c} = {\color{red}a} \cdot ({\color{blue}b} - {\color{blue}c}) \end{aligned}$$

We can rewrite $6x^2 - 3x$ as a difference of two products. Afterwards, we can rewrite an equivalent expression using the distributive property. We call this process **factoring out the gcf**.

Definition (The #1 Rule of Factoring)

The first step to factoring any algebraic expression is to factor out the gcf (if there is one).

Definition

The **greatest common factor (GCF)** for a polynomial is the largest monomial that divides (is a factor of) each term of the polynomial.

Example: The greatest common factor for $25x^5 + 20x^4 - 30x^3$ is $5x^3$ since it is the largest monomial that is a factor of each term.

$$\begin{aligned} 25x^5 + 20x^4 - 30x^3 &= 5x^3 \cdot (5x^2) + 5x^3 \cdot (4x) - 5x^3 \cdot (6) \\ &= 5x^3 \cdot (5x^2 + 4x - 6) \end{aligned}$$

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1. Find the GCF of the coefficients of each variable factor.

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2. For each variable factor common to all the terms, determine the smallest exponent that the variable factor is raised to.
3. Compute the product of the common factors found in Steps 1 and 2. This expression is the GCF of the polynomial.

Factor the greatest common factor from each of the following.

- $8x^3 - 8x^2 - 48x$
- $15a^7 - 25a^5 + 30a^3$
- $12x^4y^5 - 9x^3y^4 - 15x^5y^3$
- $4(a+b)^4 + 6(a+b)^3 + 16(a+b)^2$
- $x(x+7) + 2(x+7)$

Factoring Trinomials with a Leading Coefficient of 1

Earlier in the chapter, we multiplied binomials.

$$(x+2)\cdot(x+8) = x^2 + 10x + 16$$

$$(x+6)(x+3) = x^2 + 9x + 18$$

In each case, the product of the two binomials is a trinomial.

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$$(x+6)(x+3)=x^2+9x+18$$

In each case, the product of the two binomials is a trinomial. The first term in the resulting trinomial is obtained by multiplying the first term in each binomial. The middle term arises from adding the product of the two inside terms with the product of the two outside terms. The last term is the product of the two outside terms. The last term is the product of the last terms in each binomial.

Factoring Trinomials with a Leading Coefficient of 1

In general,

$$(x+a)\cdot(x+b)=x^2+ax+bx+a\cdot b$$

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$$x^2 + (a+b)x + a\cdot b = (x+a)\cdot(x+b)$$

To factor a trinomial with a leading coefficient of 1, we simply find the two numbers *a* and *b* whose sum is the coefficient of the middle term, and whose product is the constant term.

Factoring Trinomials with a Leading Coefficient of 1

Factor.

- $x^2 + 5x + 4$
- $x^2 + 7x + 6$
- $x^2 + 9x + 14$
- $x^2 + 11x + 24$
- $x^2 + 19x + 34$
- $x^2 + 12x + 27$
- $x^2 + 20x + 64$
- $x^2 + 18x + 65$
- $x^2 - x + 5$
- $x^2 + 5xy + 4y^2$
- $x^2 + 5xy + 6y^2$
- $x^2 + 12xy + 27y^2$
- $m^2 + 19mn + 60n^2$
- $x^2 + 2x - 15$
- $x^2 - 7x - 18$
- $x^2 + x - 20$
- $x^2 + 10x - 24$

Polynomials with four terms can sometimes be **factored by grouping**.

Example Factor $x^4 - 2x^3 - 8x + 16$

Solution:

$$x^4 - 2x^3 - 8x + 16 = (x^4 - 2x^3) + (-8x + 16) \quad \text{(assoc. prop +)}$$

$$= \left[x \cdot (x^3) + (-2) \cdot (x^3)\right] + \left[(-8) \cdot x + (-8) \cdot (-2)\right]$$

$$= x^3(x - 2) - 8(x - 2) \quad \text{(distr. prop.)}$$

$$= x^3(x - 2) - 8(x - 2) \quad \text{(identify common factor)}$$

$$= \boxed{(x - 2)(x^3 - 8)} \quad \text{(distr. prop)}$$

**Technically we are not done. This is not the prime factorization of the given polynomial since $x^3 - 8$ can be factored with the difference of cubes formula.

Problem: Factor $ax^2 + bx + c$	Problem: Factor $10x^2 - 11x - 6$
(1) Multiply a times c.	(1) $a = 10$, $b = -11$, $c = -6$, so clearly $a \cdot c = -60$
(2) List all possible pairs of numbers whose product is ac	-60 $\qquad$ -60 $\swarrow\searrow$ $\qquad$ $\swarrow\searrow$ $-6 \cdot 10$ $\qquad$ $-1 \cdot 60$ $6 \cdot (-10)$ $\qquad$ $1 \cdot (-60)$ $3 \cdot (-20)$ $\qquad$ $-5 \cdot 12$ $-3 \cdot 20$ $\qquad$ $5 \cdot (-12)$ $\boxed{-15 \cdot 4}$ $\qquad$ $15 \cdot (-4)$
(3) Box the pair whose sum is $b \nearrow$	(3) $b = -11$, and $-15 + 4 = -11$

Problem: Factor $ax^2 + bx + c$	Problem: Factor $10x^2 - 11x - 6$
(4) Replace *b* with the sum of the circled pair. Distribute *x* into this quantity	(4) $10x^2 - 11x - 6$ $= 10x^2 + (-15x + 4x) - 6$ $= 10x^2 - 15x + 4x - 6$
(5) Now factor by grouping: Use parenthesis to group the first two terms, and another () to group the second two terms.	(5) $(10x^2 - 15x) + (4x - 6)$ $= 5x(2x - 3) + 2(2x - 3)$ $= 5x(2x - 3) + 2(2x - 3)$ $= (2x - 3) \cdot (5x + 2)$

Recall: The Number Types

Definition

The set of whole numbers,

$$\mathbb{W} = \{0, 1, 2, 3, 4, \dots\}$$

is the set of natural numbers unioned with zero, written $\mathbb{W} = \mathbb{N} \cup \{0\}$.

Recall: The Number Types

Definition

The set of integers,

$$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

is also known as all the positive and negative whole numbers.

Recall: The Number Types

Definition

A rational number is any number that can be expressed as the ratio of two integers. The set of rational numbers is written symbolically as

$$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; \text{a and b are any integers, and } b \neq 0 \right\}$$

Note that any integer "a" is a rational number since $a = \frac{a}{1}$.

Rational expressions

A rational expression is defined similarly as any expression that can be written as the ratio of two polynomials.

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$$\text{rational expressions} = \left\{ \frac{p}{q} \;\middle|\; \text{p and q are polynomials, } \; q \neq 0 \right\}$$

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Definition (Rational Expressions)

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Some examples of rational expressions are

$$\frac{1}{x}, \qquad \frac{2m-3}{6n-7}, \qquad \frac{x^2-3x-1}{x^2-3x-5}, \qquad \frac{x-y}{y-x}$$

Rational expressions

Basic Properties

Multiplying (or dividing) the numerator and denominator by the same nonzero expression may change the form of the rational expression, but it will always produce an expression equivalent to the original one.

Rational expressions

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Multiplying (or dividing) the numerator and denominator by the same nonzero expression may change the form of the rational expression, but it will always produce an expression equivalent to the original one.

We use this property to reduce fractions to lowest terms. For example,

$$\frac{6}{8} = \frac{3 \cdot \cancel{2}}{4 \cdot \cancel{2}} = \frac{3}{4}$$

Using Basic Properties

In a similar fashion, we reduce rational expressions to lowest terms by

1. first factoring the numerator and denominator,
2. and then dividing both numerator and denominator by any factors they have in common.

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Example: Reduce $\frac{x^2 - 25}{x - 5}$ to lowest terms.

Solution:

$$\frac{x^2 - 25}{x - 5} = \frac{(x-5)\cdot(x+5)}{x-5} = \frac{\cancel{(x-5)}\cdot(x+5)}{\cancel{(x-5)}} = x + 5$$

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Solution:

$$\frac{x-5}{x^2-10x+25}=\frac{x-5}{(x-5)^2}=\frac{1\cdot(x-5)}{(x-5)\cdot(x-5)}=\frac{1\cdot\cancel{(x-5)}}{(x-5)\cancel{(x-5)}}=\frac{1}{x-5}$$

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Solution:

$$\frac{-3+5x}{25x^2-9}=\frac{5x-3}{(5x)^2-3^2}=\frac{1\cdot(5x-3)}{(5x+3)\cdot(5x-3)}=\frac{1\cdot\cancel{(5x-3)}}{(5x+3)\cancel{(5x-3)}}=\frac{1}{5x+3}$$

Try This One! Reduce $\dfrac{5x-3}{3-2x}$ to lowest terms.

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Solution:

First degree polynomials have form $ax + b$ for real numbers a and b with a not equal to zero. First degree polynomials are always prime, unless the numbers a and b have a greatest common factor. So, the given expression is prime (not factorable), since both first degree polynomials do not have a common constant that can be divided out of both numerator and denominator. Therefore, the given rational expression is in lowest terms.

Try This One! Reduce $\dfrac{16y^3 - 250}{12y^2 - 26y - 10}$ to lowest terms.

Try This One! Reduce $\dfrac{16y^3-250}{12y^2-26y-10}$ to lowest terms.

Solution: $\dfrac{16y^3-250}{12y^2-26y-10}=\dfrac{\cancel{2}\cdot(8y^3-125)}{\cancel{2}\cdot(6y^2-13y-5)}=\dfrac{(2y)^3-5^3}{6y^2+2y-15y-5}$

Try This One! Reduce $\frac{16y^3-250}{12y^2-26y-10}$ to lowest terms.

Solution: $$\frac{16y^3-250}{12y^2-26y-10} = \frac{\cancel{2}\cdot(8y^3-125)}{\cancel{2}\cdot(6y^2-13y-5)} = \frac{(2y)^3-5^3}{6y^2+2y-15y-5}$$

$$= \frac{(2y-5)(4y^2+10y+25)}{(6y^2+2y)+(-15y-5)} = \frac{(2y-5)(4y^2+10y+25)}{2y\cdot(3y+2)+(-5)\cdot(3y+2)} = \frac{\cancel{(2y-5)}(4y^2+10y+25)}{(3y+2)\cdot\cancel{(2y-5)}}$$

$$= \frac{(4y^2+10y+25)}{(3y+2)}$$

Try This One! Reduce $\dfrac{3a^3+3}{6a^2-6a+6}$ to lowest terms.

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Solution: $\frac{3a^3+3}{6a^2-6a+6} = \frac{3(a^3+1)}{6(a^2-a+1)} = \frac{3(a+1)\cancel{(a^2-a+1)}}{6\cancel{(a^2-a+1)}}$

$$= \frac{3(a+1)}{6} = \frac{3(a+1)}{3\cdot 2} = \frac{\cancel{3}(a+1)}{\cancel{3}\cdot 2} = \frac{(a+1)}{2}$$

Try This One! Reduce $\dfrac{x^2-3x+ax-3a}{x^2-ax-3x+3a}$ to lowest terms.

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Solution:

$$\frac{x^2-3x+ax-3a}{x^2-ax-3x+3a}=\frac{(x^2-3x)+(ax-3a)}{(x^2-ax)+(-3x+3a)}=\frac{x(x-3)+a(x-3)}{x(x-a)+(-3)(x-a)}$$

$$=\frac{(x+a)(x-3)}{(x-a)(x-3)}=\frac{(x+a)\cancel{(x-3)}}{(x-a)\cancel{(x-3)}}=\frac{(x+a)}{(x-a)}$$

Try This One! Reduce $\dfrac{a-b}{b-a}$ to lowest terms.

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