Chapter 2

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The graph of a function f is the graph of the equation $y = f(x)$. A function is called **continuous** if its graph has no breaks or holes.

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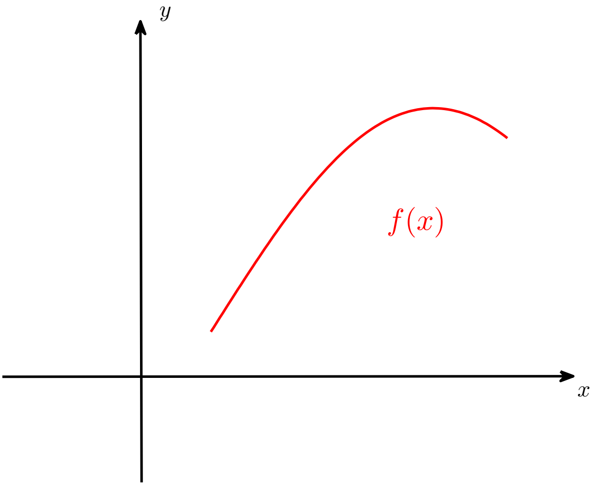

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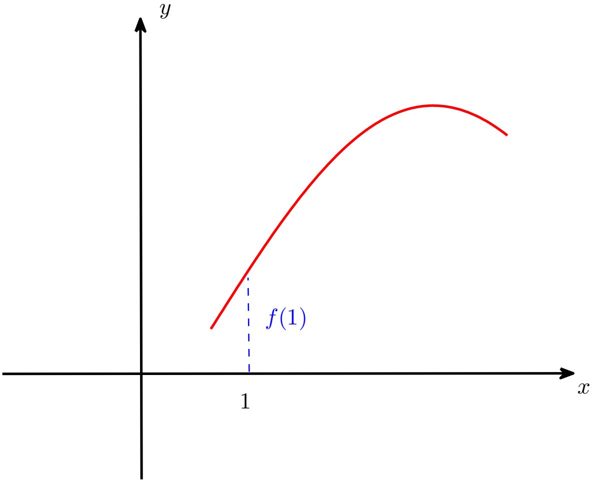

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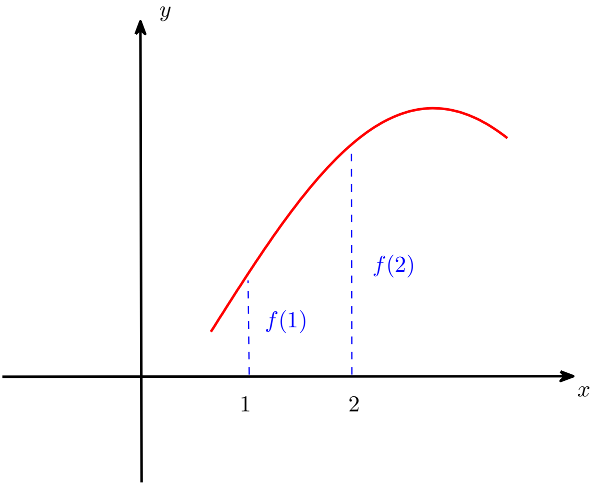

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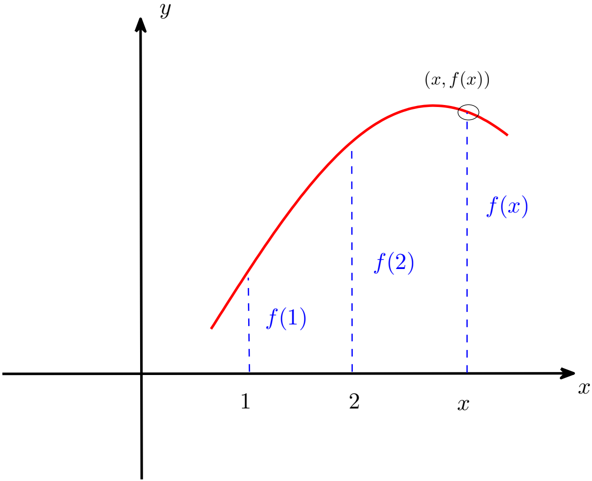

Definition

A function is a special type of relation. A <u>FUNCTION</u> is a correspondence between a first set, called the *domain*, and a second set, called the *range*, such that each member of the domain corresponds to *exactly one* member of the range.

However, different elements of the domain are allowed to have a correspondence with the same value in the range.

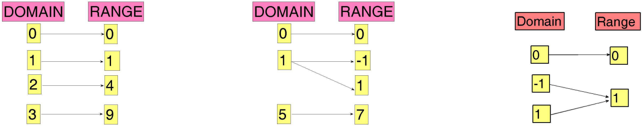

Figure : F is a FUNCTION (left), R is a relation but NOT A FUNCTION (center) & an example of a function (right) whose two different domain elements are associated with the same range element.

Theorem (VERTICAL LINE TEST (VLT))

A curve in the coordinate plane is the graph of a function if and only if there is no vertical line that crosses the graph more than once.

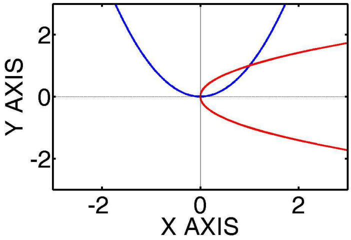

Figure : GRAPHS OF $y = x^2$ and $x = y^2$

Does the equation $x^2 + y^2 = 16$ define y as a function of x?

$p(x) = x^n$ is called a **power function**.

- If n is even, the graph of $f(x) = x^n$ is similar to the parabola $y = x^2$.
- If n is odd, the graph of $f(x) = x^n$ is similar to the cubic $y = x^3$.

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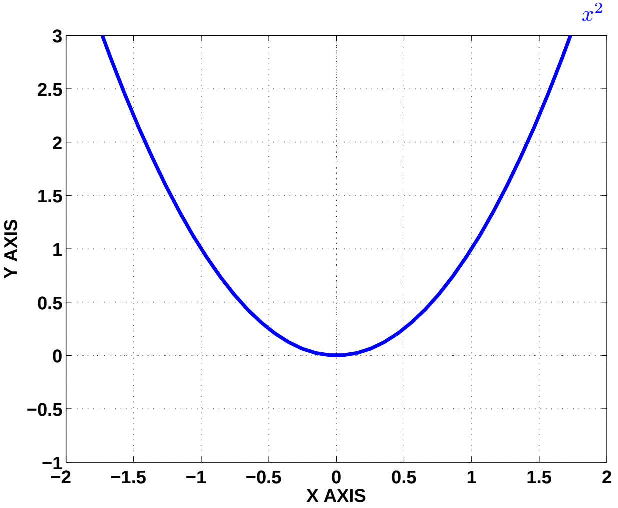

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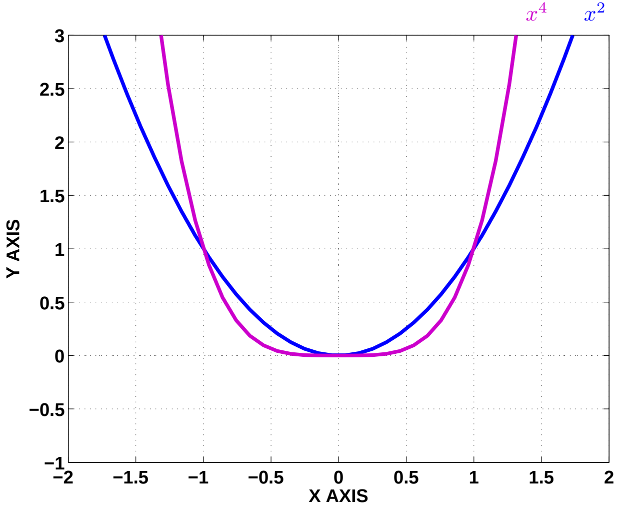

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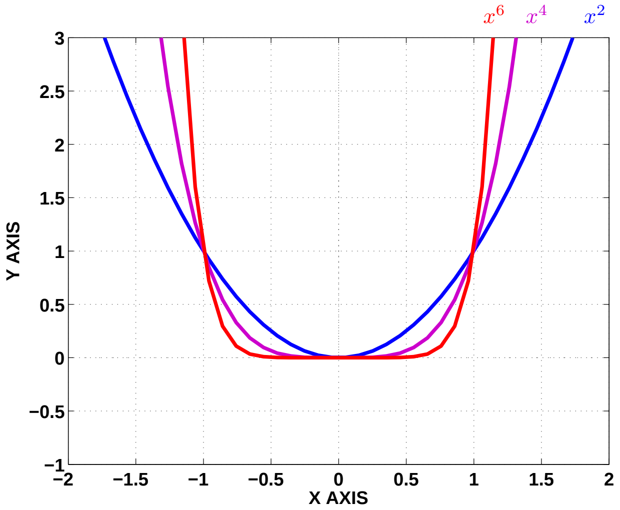

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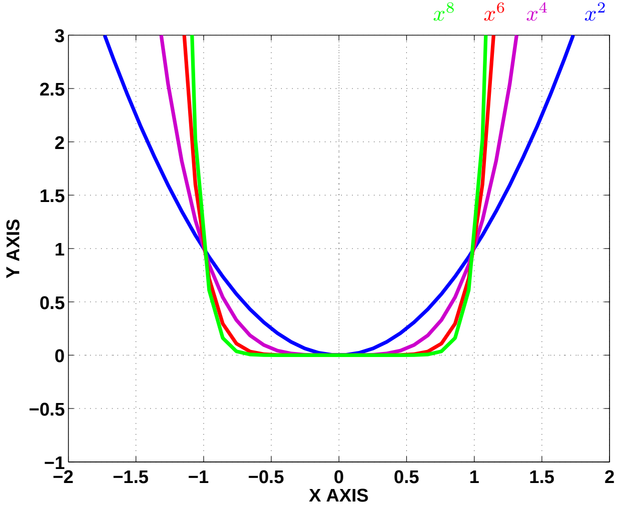

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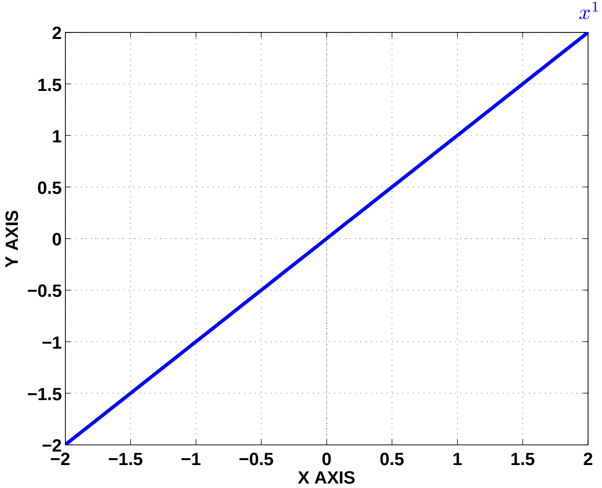

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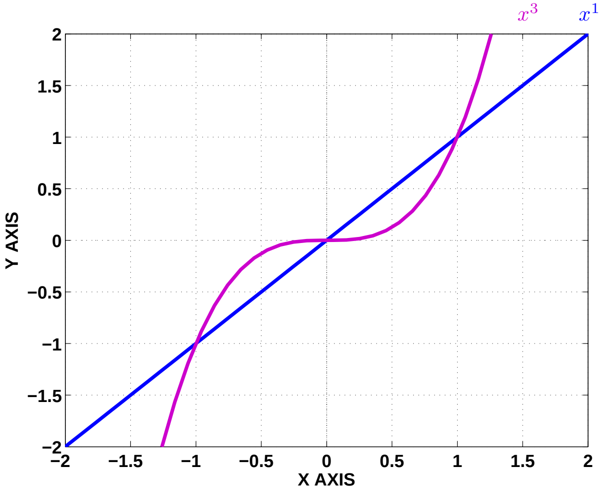

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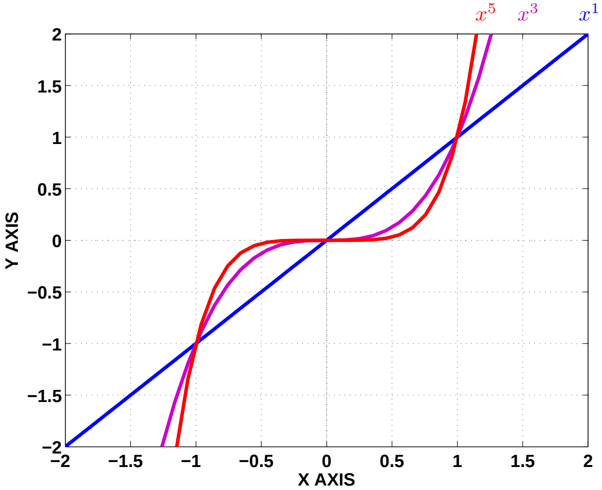

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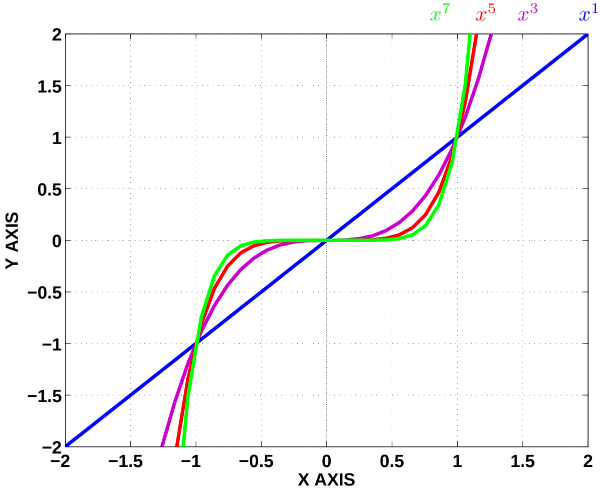

$f(x) = \dfrac{1}{x^n}$ is called a **reciprocal function**.

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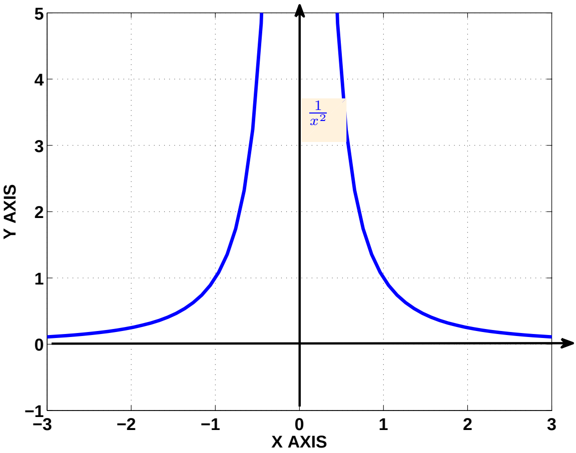
n even



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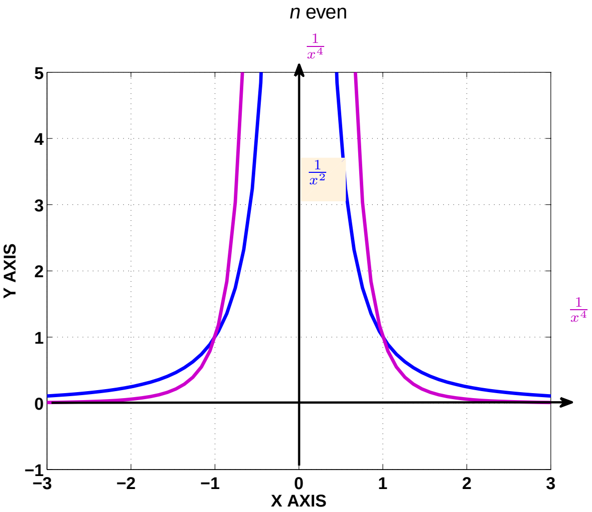

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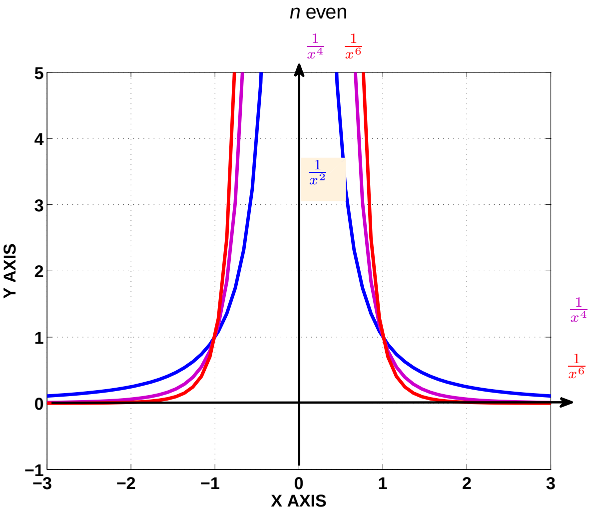

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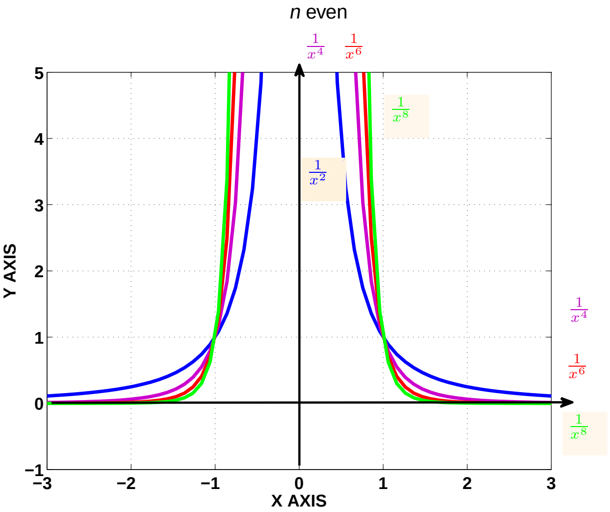

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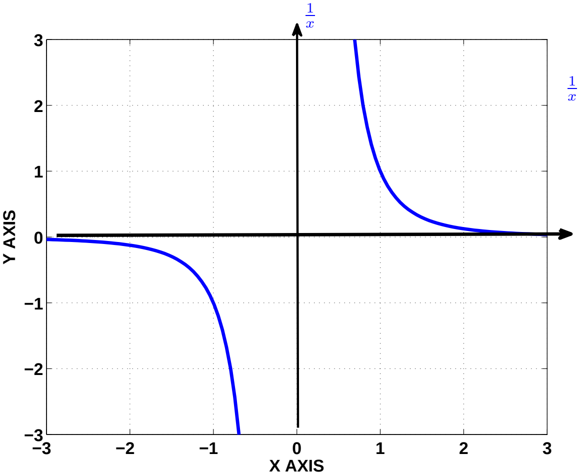

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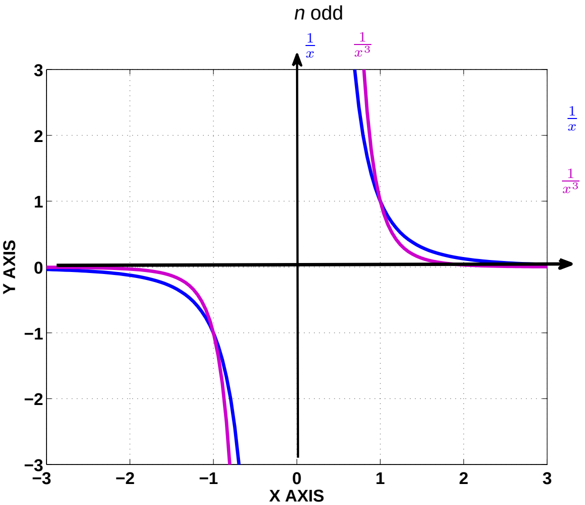

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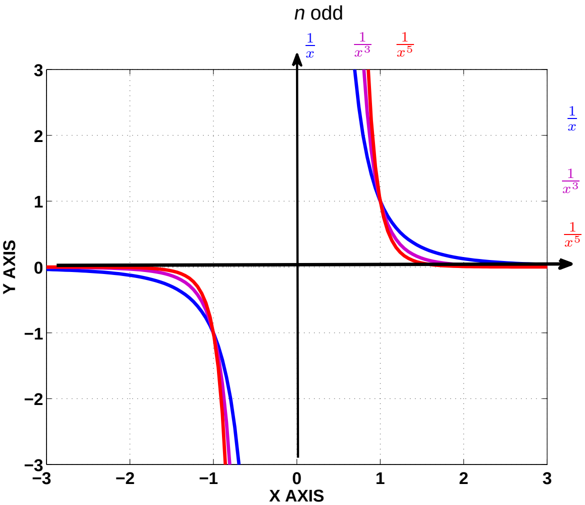

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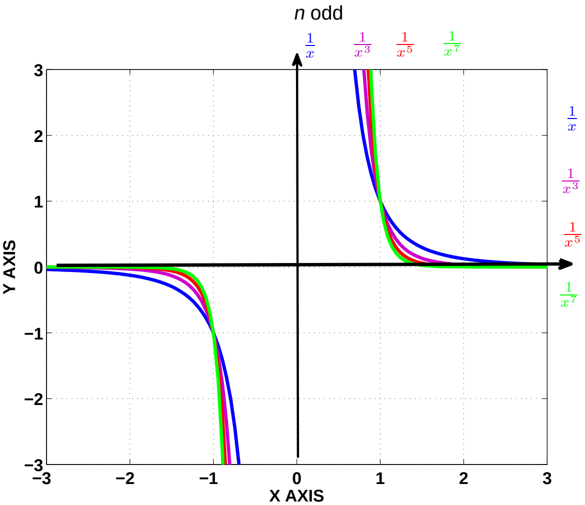

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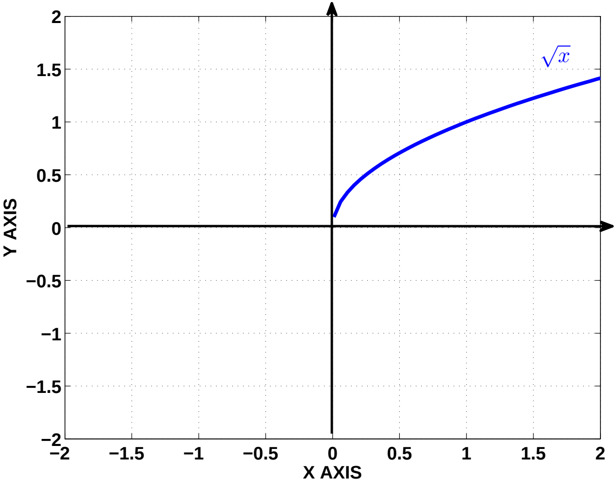

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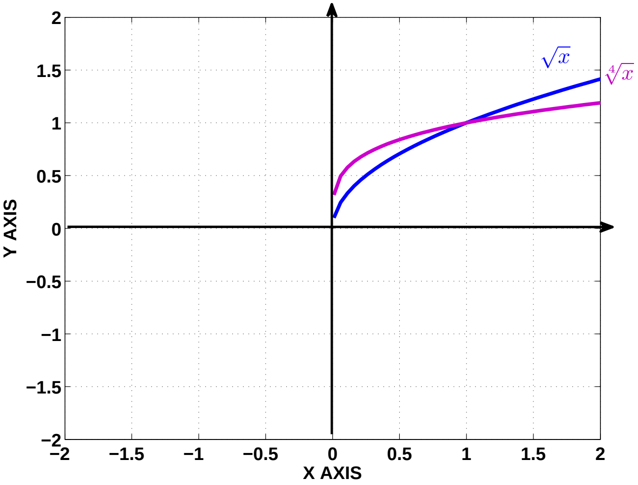

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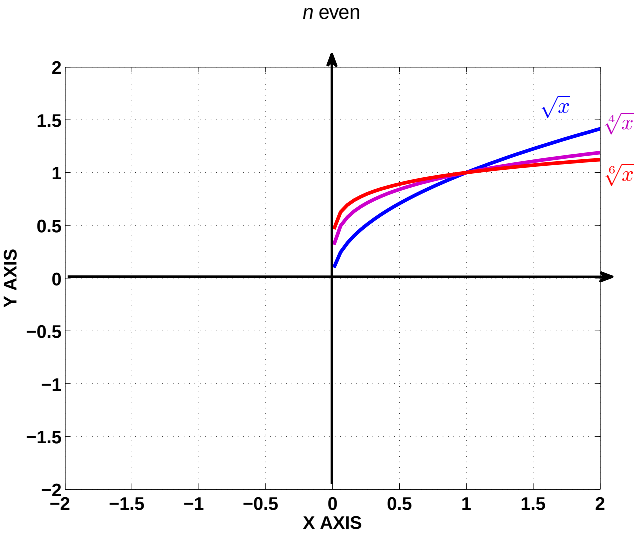

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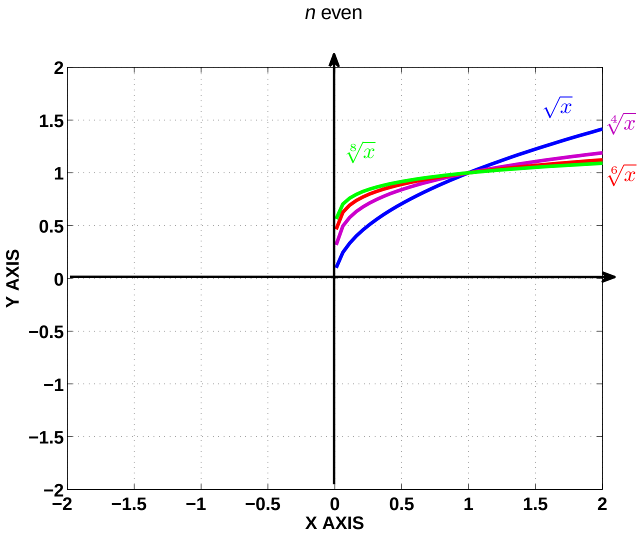

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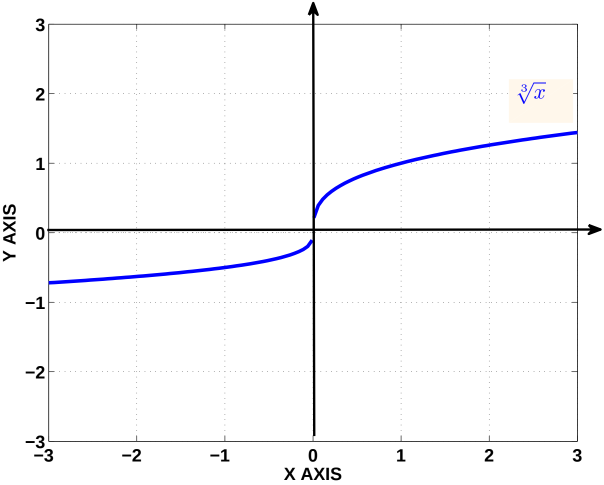

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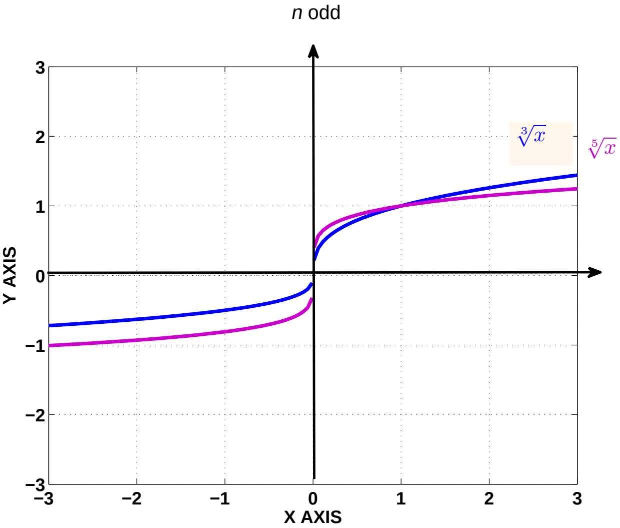

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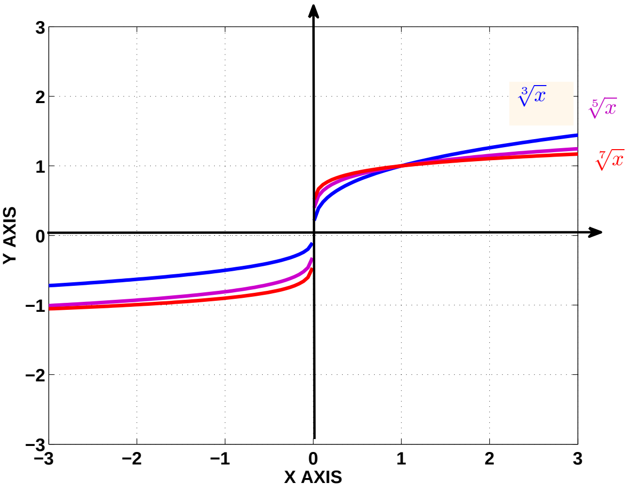

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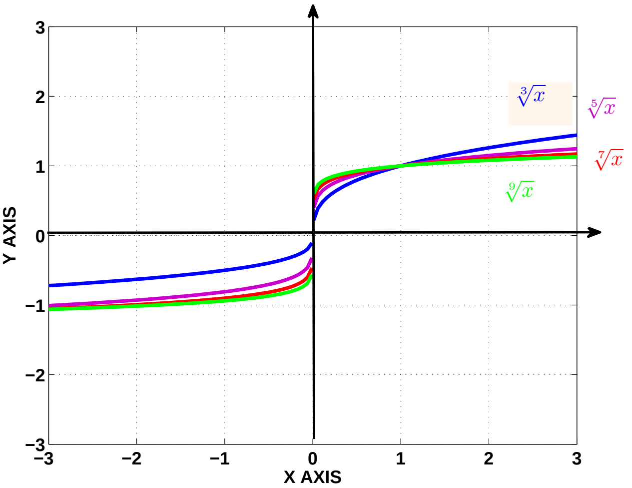
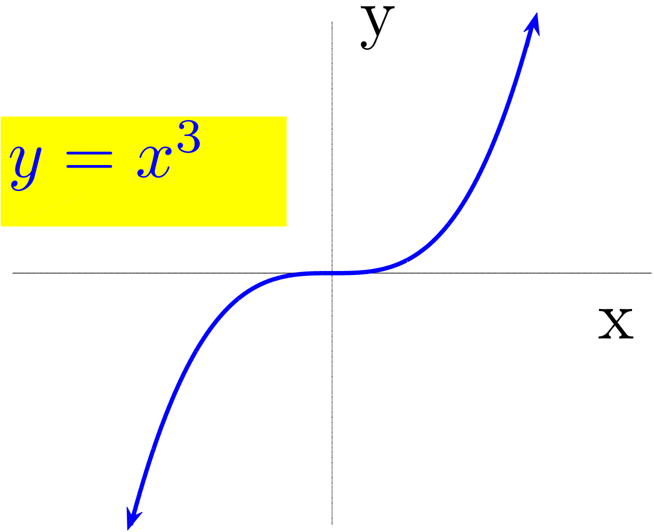
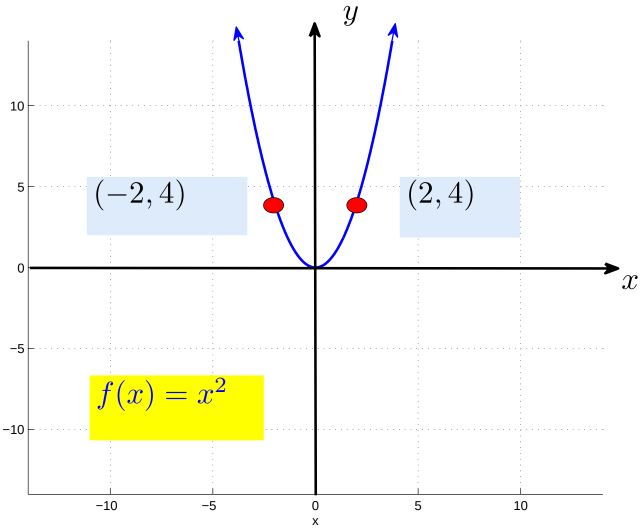


The graph of a function has origin symmetry when for any point (x,y) on the graph, there is also a point (-x,-y) on the graph.

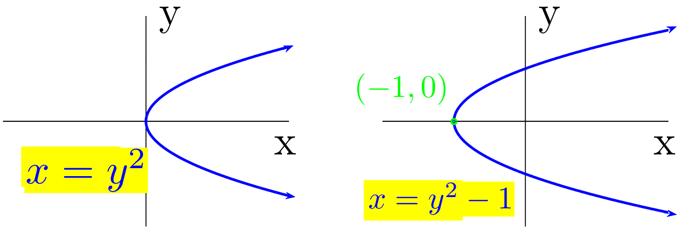


The graph of a function has y-axis symmetry if for every point (x,y), there is also a point (-x,y) on the graph.



Definition

The graph of a relation has x-axis symmetry if for every point (x, y) on the graph, the point $(x, -y)$ is also on the graph.



Can a function have x-axis symmetry?

Definition

A function $f(x)$ can be classified as (one of the following):

1. Even
2. Odd
3. Neither Even Nor Odd

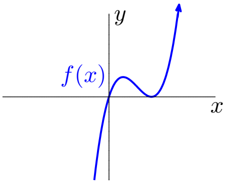


Figure : A function that is neither: $f(x) = x(x-2)^2$

How to Test for Symmetry

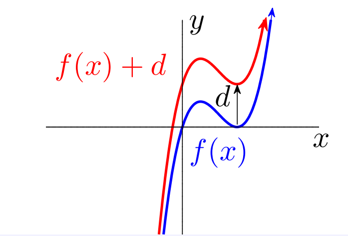
- A function is EVEN if its graph has y-axis symmetry. If substitution of $-x$ for x leads to the same equation, i.e., **If $f(-x) = f(x)$, then f is an even function.**

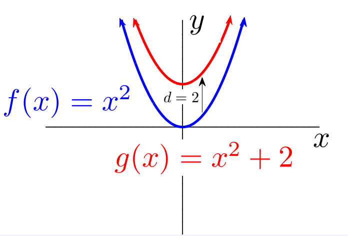
- A function is ODD if its graph has origin symmetry. If substitution of $-x$ for x leads to the negative version of f, i.e.,
  If $f(-x) = -f(x)$, then f is an odd function.

If a, b, c, and d are real numbers with $a \neq 0$, then $y = a \cdot f(bx - c) + d$ is called a **linear transformation** of the function $y = f(x)$.

Vertical Shifts of Graphs

Suppose $d > 0$. The graph of $y = f(x) + d$ is the graph of $y = f(x)$ shifted vertically *upward* d units.

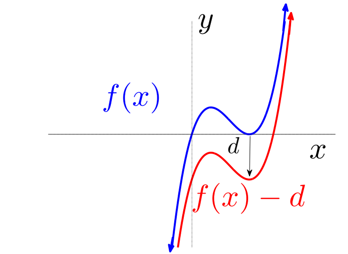


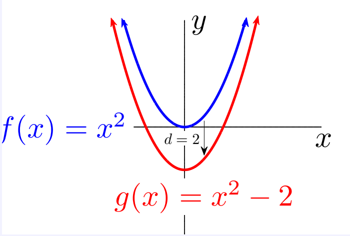


The graphs of $f(x) = x(x-2)^2$ and $f(x) + d$ (left); and the graphs of $f(x) = x^2$ and $g(x) = x^2 + 2$ (right).

Vertical Shifts of Graphs

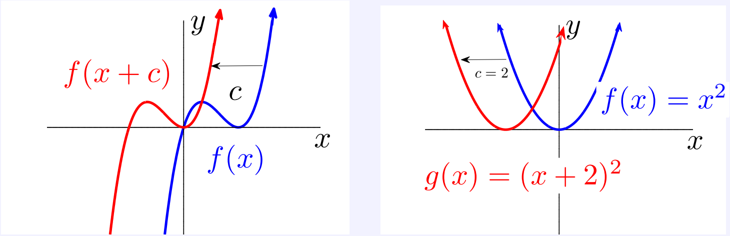
Suppose $d > 0$. The graph of $y = f(x) - d$ is the graph of $y = f(x)$ shifted vertically *downward* d units.





Horizontal Shifts of Graphs

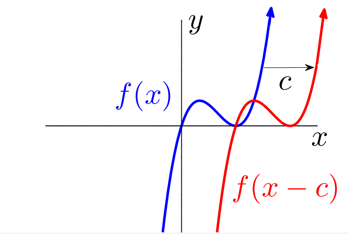
Let $c > 0$. The graph of $y = f(x + c)$ is the graph of $y = f(x)$ shifted to the *left* c units.

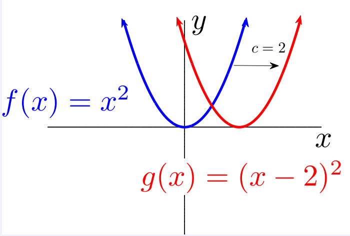


The graphs of $f(x) = x(x-2)^2$ and $f(x + c)$ are given in the left panel; and the graphs of $f(x) = x^2$ and $g(x) = (x + 2)^2$ are presented in the right panel above.

Horizontal Shifts of Graphs

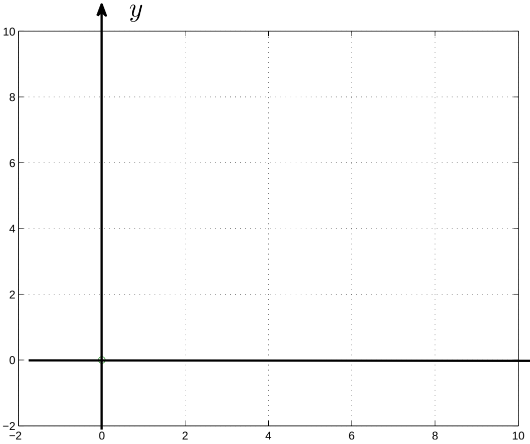
Let $c > 0$. The graph of $y = f(x-c)$ is the graph of $y = f(x)$ shifted to the *right* c units.





Combining Horizontal and Vertical Shifts

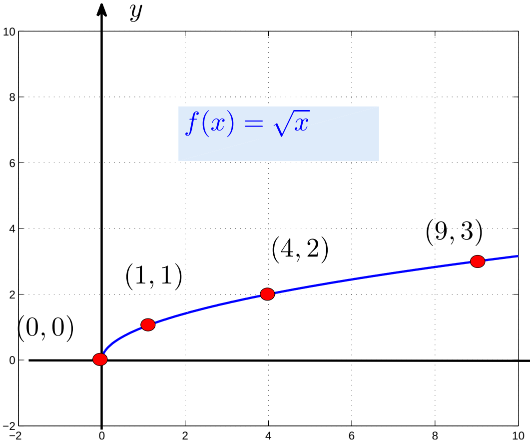
Graph $y = \sqrt{x-4} + 3$



Combining Horizontal and Vertical Shifts

Graph $y = \sqrt{x-4} + 3$

1.) $f(x) = \sqrt{x}$

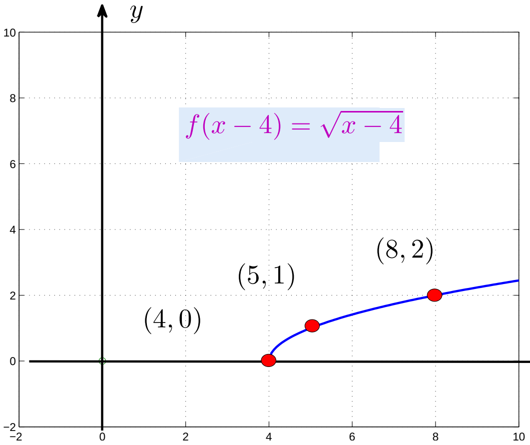


Combining Horizontal and Vertical Shifts

Graph $y = \sqrt{x-4} + 3$

1.) $f(x) = \sqrt{x}$

2.) Horizontal Shift



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Graph $y = \sqrt{x-4} + 3$

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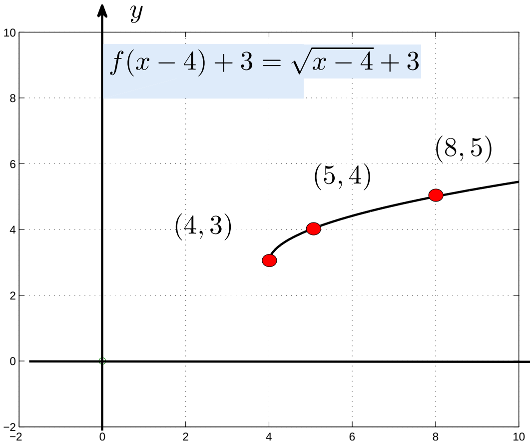
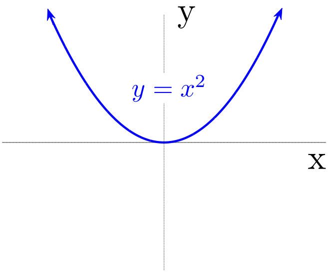
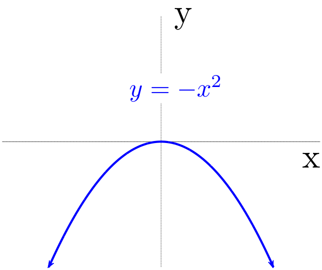
2.) Horizontal Shift

3.) Vertical Shift

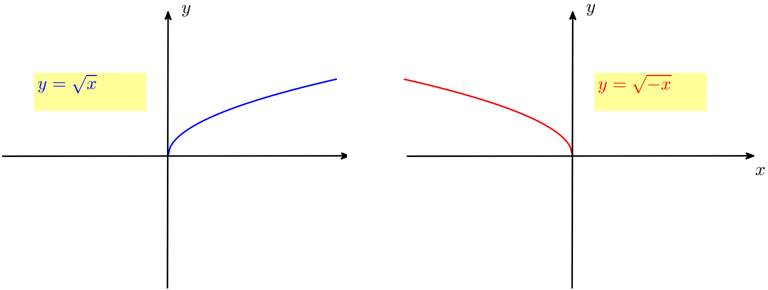


To graph $y = -f(x)$ **reflect** the graph of $f(x)$ about the x-axis.





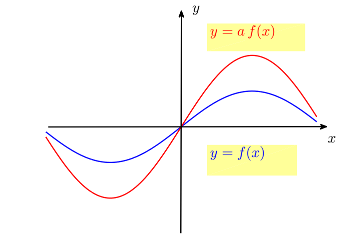
To graph $y = f(-x)$ **reflect** the graph of $f(x)$ about the y-axis.



Vertical Stretching and Shrinking

Graphing $y = a \cdot f(x)$

If $a > 1$, stretch the graph of $y = f(x)$ vertically by a factor of a.



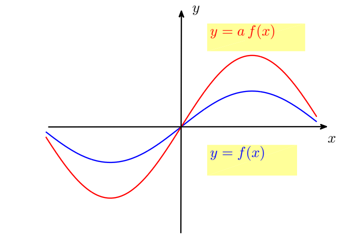
$a > 1$

Vertical Stretching and Shrinking

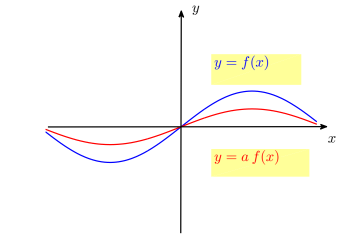
Graphing $y = a \cdot f(x)$

If $a > 1$, stretch the graph of $y = f(x)$ vertically by a factor of a.

If $0 < a < 1$, shrink the graph of $y = f(x)$ vertically by a factor of a.



$a > 1$



$0 < a < 1$

Example: $f(x) = a \cdot x^2$

If $a > 1$, stretch the graph of $y = f(x)$ vertically by a factor of a.

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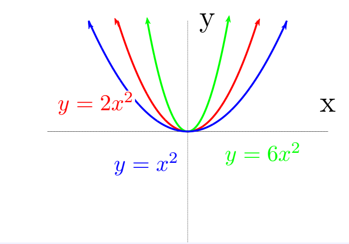

$a > 1$

Example: $f(x) = a \cdot x^2$

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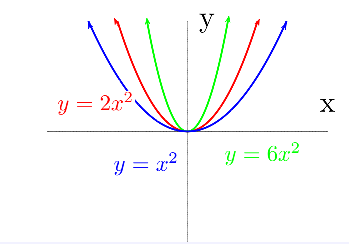
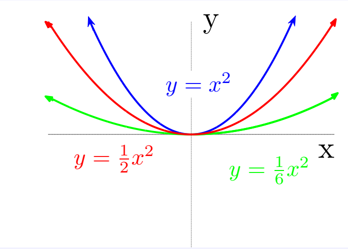
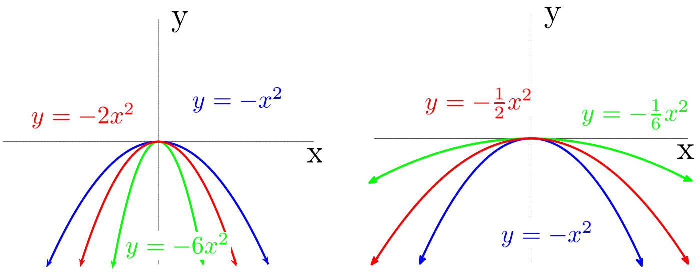


$a > 1$



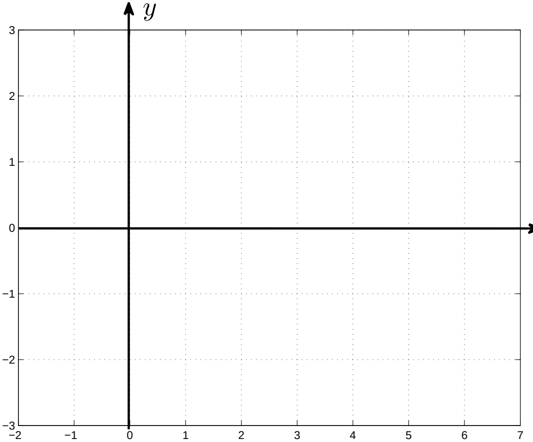
$0 < a < 1$

Reflection and Vertical Shrinking & Stretching $y = x^2$



Combining Shifting, Stretching and Reflecting

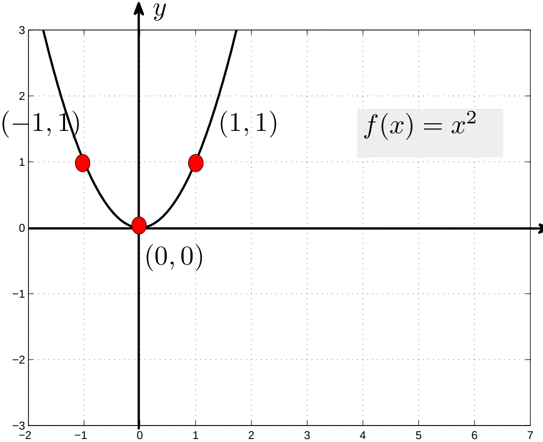
Graph $y = 1 - 2(x - 3)^2$



Combining Shifting, Stretching and Reflecting

Graph $y = 1 - 2(x - 3)^2$

1.) $f(x) = x^2$

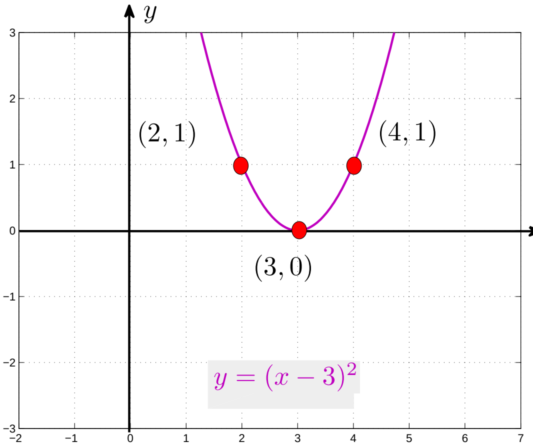


Combining Shifting, Stretching and Reflecting

Graph $y = 1 - 2(x - 3)^2$

1.) $f(x) = x^2$

2.) Horizontal Shift



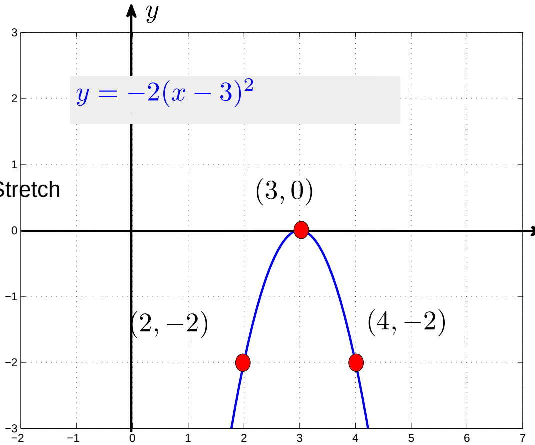
Combining Shifting, Stretching and Reflecting

Graph $y = 1 - 2(x - 3)^2$

1.) $f(x) = x^2$

2.) Horizontal Shift

3.) Reflection and Vertical Stretch



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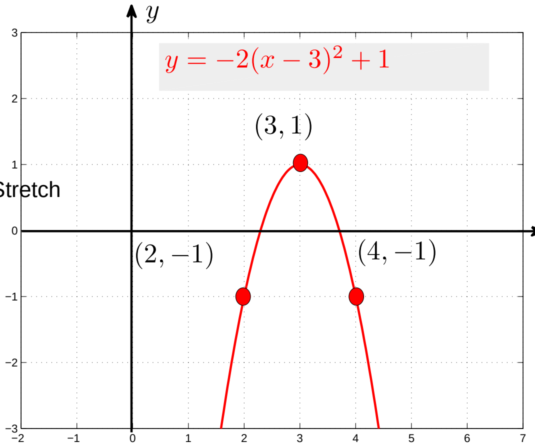
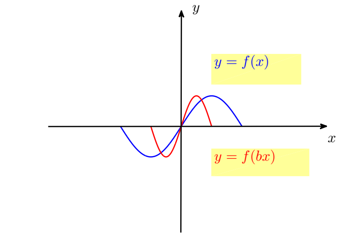


Horizontal Stretching and Shrinking

Graphing $y = f(b \cdot x)$

If $b > 1$, shrink the graph of $y = f(x)$ horizontally by a factor of $1/b$.



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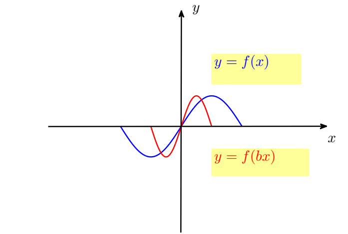
Horizontal Stretching and Shrinking

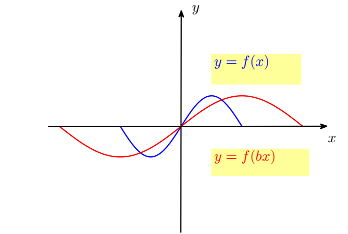
Graphing $y = f(b \cdot x)$

If $b > 1$, shrink the graph of $y = f(x)$ horizontally by a factor of $1/b$.

If $0 < b < 1$, stretch the graph of $y = f(x)$ horizontally by a factor of $1/b$.

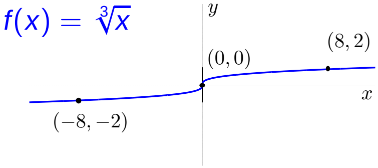


$b > 1$

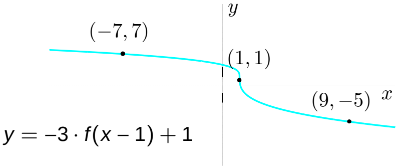


$0 < b < 1$

If a, b, c, and d are real numbers with $a \neq 0$, then $y = a \cdot f(bx - c) + d$ is called a **linear transformation** of $y = f(x)$.



All of the transformations of a function **form a family of functions**. For example, $y = -3\sqrt[3]{x-1} + 1$ (graph below) is in the cube root family of functions.



Linear Transformation

$$y = a \cdot f(bx - c) + d$$

a represents the reflection and vertical shrinking or stretching of f.

b represents the horizontal shrinking or stretching of f.

c represents the horizontal translation of f.

d represents the vertical translation of f.

Graphing using Translations

$$y = a \cdot f(bx - c) + d$$

1.) Identify and graph $f(x)$. Use symmetry, if possible.

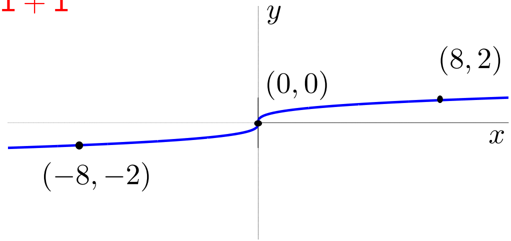
2.) Horizontal Shift

3.) Reflection *and* horizontal and/or vertical shrinking or stretching

4.) Vertical Shift

Graph $y = -3 \cdot \sqrt[3]{x-1} + 1$

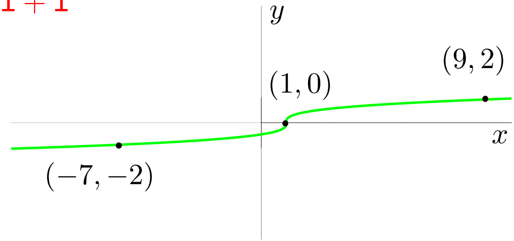
Graph $y = -3 \cdot \sqrt[3]{x-1} + 1$



1.) $f(x) = \sqrt[3]{x}$

x	$f(x) = \sqrt[3]{x}$
−8	−2
−1	−1
0	0
1	1
8	2

Graph $y = -3 \cdot \sqrt[3]{x-1} + 1$

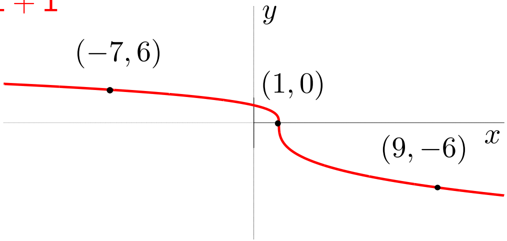


1.) $f(x) = \sqrt[3]{x}$

2.) Horizontal Shift

x	$y = \sqrt[3]{x-1}$
−7	−2
0	−1
1	0
2	1
9	2

Graph $y = -3 \cdot \sqrt[3]{x-1} + 1$



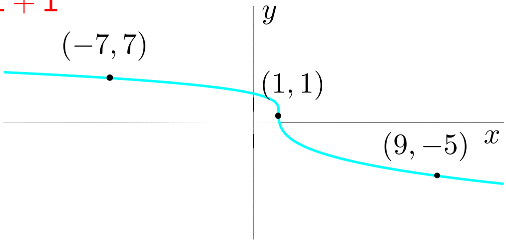
1.) $f(x) = \sqrt[3]{x}$

2.) Horizontal Shift

3.) Reflection and Vertical Stretching

x	$y = \sqrt[3]{x-1}$	$y = -3\sqrt[3]{x-1}$
−7	−2	6
0	−1	3
1	0	0
2	1	−3
9	2	−6

Graph $y = -3 \cdot \sqrt[3]{x-1} + 1$



1.) $f(x) = \sqrt[3]{x}$

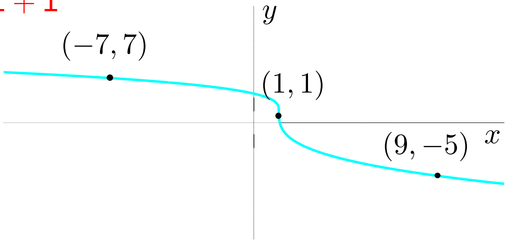
2.) Horizontal Shift

3.) Reflection and Vertical Stretching

4.) Vertical Shift

x	$y = -3\sqrt[3]{x-1} + 1$
−7	7
0	4
1	1
2	−2
9	−5

Graph $y = -3 \cdot \sqrt[3]{x-1} + 1$



1.) $f(x) = \sqrt[3]{x}$

2.) Horizontal Shift

3.) Reflection and Vertical Stretching

4.) Vertical Shift

x	$y = -3\sqrt[3]{x-1} + 1$
-7	7
0	4
1	1
2	-2
9	-5

$y = -3 \cdot \sqrt[3]{x-1} + 1 = -3 \cdot f(x-1) + 1$
has the following characteristics:

- domain: $x \in (-\infty, \infty)$
- range: $y \in (-\infty, \infty)$
- $g(x)$ is a decreasing function.
  $g(x) \downarrow$ for $x \in (-\infty, \infty)$

Set Intersection

The intersection of two sets A and B, written $A \cap B$, is the set of all elements (numbers) that are in both A and B. The $\cap$ symbol means the word "and."

Example: Suppose $A = \{1,2,3,4\}$ and $B = \{2,4,20\}$. Then $A \cap B = \{2,4\}$

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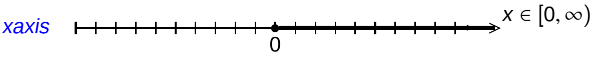
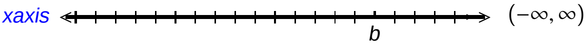
Example: $A = [0, \infty)$ and $B = (-\infty, \infty)$.





Set Intersection

The intersection of two sets A and B, written A ∩ B, is the set of all elements (numbers) that are in both A and B. The ∩ symbol means the word “and.”

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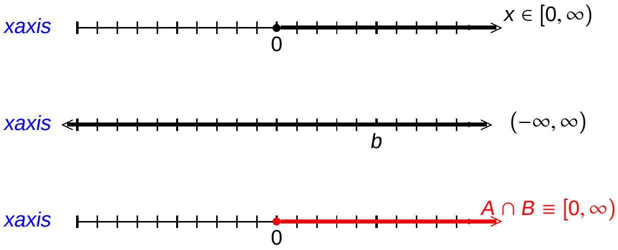
Example: A = $[0, \infty)$ and B = $(-\infty, \infty)$.



Algebra of Functions

Let f and g be functions with domains A and B. Then the functions $f+g$, $f-g$, fg, and f/g are defined as follows:

$$(f+g)(x) = f(x)+g(x) \qquad \text{domain } A \cap B$$

$$(f-g)(x) = f(x)-g(x) \qquad \text{domain } A \cap B$$

$$(f\cdot g)(x) = f(x)\cdot g(x) \qquad \text{domain } A \cap B$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \text{domain } \{x \in A \cap B \mid g(x) \neq 0\}$$

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Example: Suppose $f(x)=\sqrt{x}$, $g(x)=x^2$ and $h(x)=(f+g)(x)=\sqrt{x}+x^2$.

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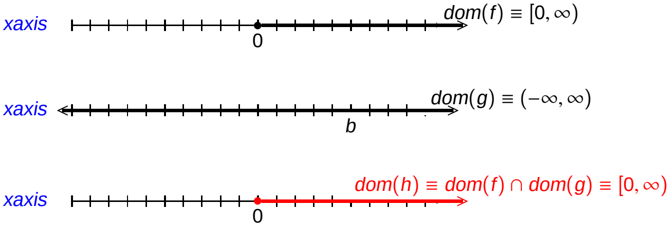

Composition of Functions

If f and g are two functions, the composition of f and g, written $f \circ g$ is defined by the equation

$$f \circ g = f(g(x)),$$

provided that $g(x)$ is in the domain of f.

Example: Suppose $f(x) = \sqrt{x}$ and $g(x) = 2x + 1$. Then $f(\ g(x)\ ) = f(2x + 1) = \sqrt{2x + 1}$.

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Example: Suppose $g \equiv \{(1,2),(3,4),(5,6)\}$ and $f \equiv \{(2,8),(4,9),(1,1)\}$. Find $f \circ g$.

Solution: Since $g(1) = 2$ and $f(2) = 8$, then $f(g(1)) = 8$, and $(1,8)$ is an ordered pair in $f \circ g$. Also since $g(3) = 4$ and $f(4) = 9$, then $f(g(3)) = 9$, and $(3,9)$ is an ordered pair in $f \circ g$. Now $g(5) = 6$ but 6 is not in the domain of f. So there are only two ordered pairs in $f \circ g$, namely $f \circ g \equiv \{(1,8),(3,9)\}$

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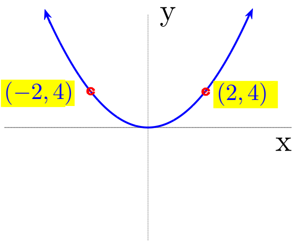
Comment: the domain of g is $\{1,3,5\}$ while the domain of $f \circ g$ is $\{1,3\}$. **In order to find the domain of $f \circ g$ we remove from the domain of g any number x such that $g(x)$ is not in the domain of f.**

One to one functions have inverses!

A function f with domain D and range R is a one to one function if *either* of the following equivalent conditions is satisfied:

Whenever $x_1 \neq x_2$ in D, then $f(x_1) \neq f(x_2)$ in R.

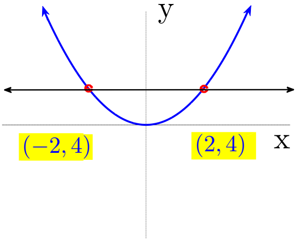
Whenever $f(x_1) = f(x_2)$ in R, then $x_1 = x_2$ in D.



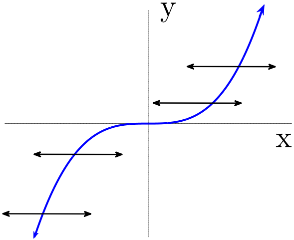
Example: $f(x) = x^2$ is *NOT* a one to one function since for $x_1 = -2$ and $x_2 = 2$, it is true that $x_1 \neq x_2$ and $f(x_1) = f(x_2) = 4$.

The Horizontal Line Test

A function f is one to one if and only if every horizontal line intersects the graph of f in at most one point.



$f(x) = x^2$
is not one to one



but $f(x) = x^3$
is one to one.

Inverse Function

Suppose f is a one to one function, with domain D and range R. The inverse function of f is the function denoted f^{-1} with domain R and range D provided that

$$f^{-1}(f(x)) = x$$

Note: A function has an inverse (function) only when it is *one to one*.

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Note: A function has an inverse (function) only when it is *one to one*.

CAUTION: $f^{-1}(x) \neq f(x)^{-1}$

- $f^{-1}(x)$ is notation for the function inverse of a one to one function f
- $f(x)^{-1} = (f(x))^{-1} = \dfrac{1}{f(x)}$ is the multiplicative inverse of the number $f(x)$.

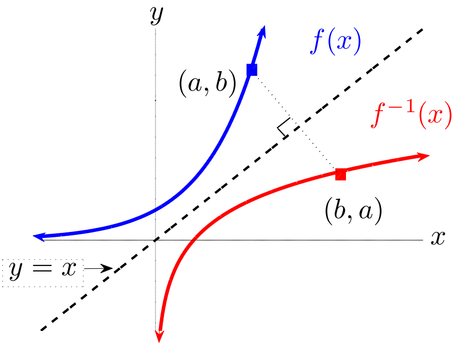
Example: Suppose f is one-to-one and $f(-9) = 15$, then $f^{-1}(15) = -9$ and $(f(-9))^{-1} = 1/15$

Properties of Inverse Functions

Suppose that f is a one to one function with domain D and range R. Then

- The inverse function f^{-1} is unique.
- The domain of f^{-1} is the range of f.
- The range of f^{-1} is the domain of f.
- The statement $f(x) = y$ is equivalent to $f^{-1}(y) = x$

Note: The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ about the line $y = x$. For every point (a, b) on the graph of $f(x)$ there is a corresponding point (b, a) on the graph of $f^{-1}(x)$.



Inverse Function

How to find the inverse of a one to one function:

1. Replace $f(x)$ with y. Then interchange x and y.
2. Solve the resulting equation for y.
3. Replace y with $f^{-1}(x)$.