5.3 The Graphs of Sine and Cosine Functions

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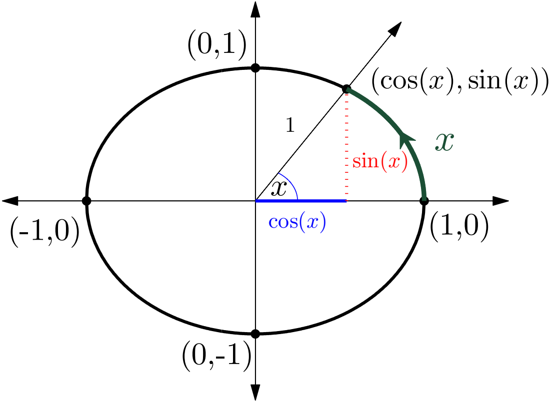
Sine function graph animation (Tim Fahlberg)
Translating Trig Graphs Applet (Guillermo Bautista)

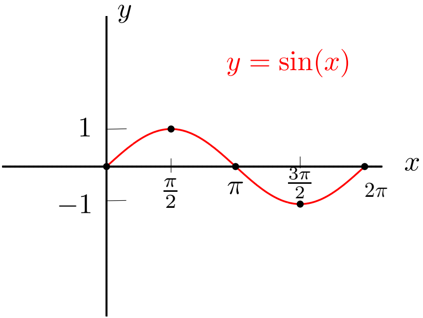
Consider an angle, x, located in standard position, such as the one given in the figure below. By definition, $\sin(x)$ is the second coordinate of the intersection of the terminal side of the angle with the unit circle, and $\cos(x)$ is the first coordinate.



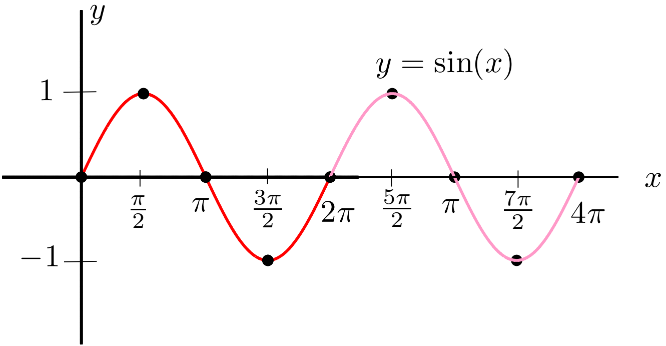
$f(x) = \sin(x)$

The graph of $y = \sin(x)$ has five key points between $x = 0$ and $x = 2\pi$, which I will refer to as quarter points.

x	$f(x)$
0	0
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	-1
2π	0

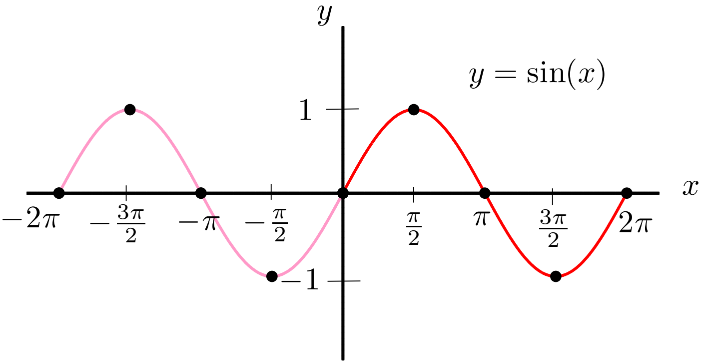


These five key points divide the x interval $[0, 2\pi]$ into *four* equal parts. Notice the x coordinates of the five key points are the 90° (or quadrantal) type angles, and the y coordinates oscillate between the maximum at one and minimum at minus one.

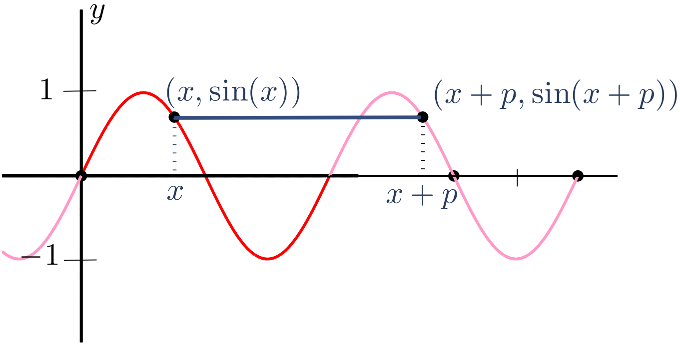


The domain of $f(x) = \sin(x)$ is all arc angles x, or real numbers x.

Therefore, the graph exists for x values outside of the interval $[0, 2\pi]$. Because $\sin(x + 2\pi) = \sin(x)$ for every x, the exact shape of the graph is repeated for $x \in [2\pi, 4\pi], [4\pi, 6\pi]$, etc.;

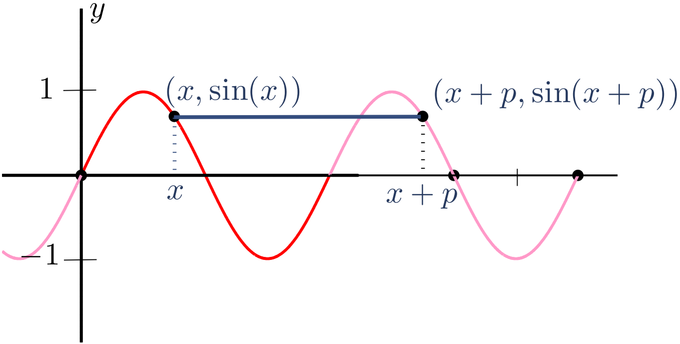


as well as for $x \in [-2\pi, 0]$, $x \in [-4\pi, -2\pi]$,... In fact the shape repeats indefinitely over the set of real numbers. Furthermore, the range of $y = \sin(x)$ is $y \in [-1, 1]$.

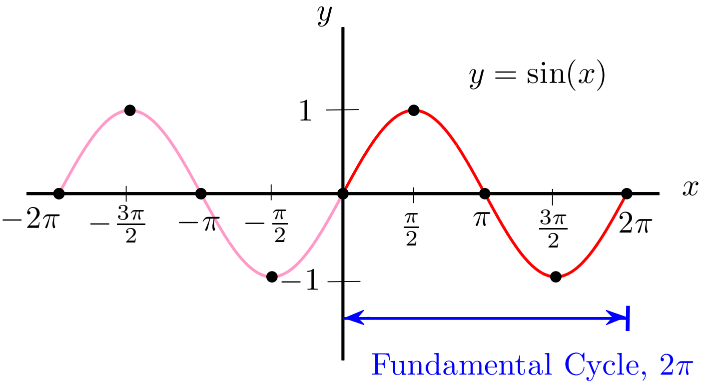


Definition (Periodic Function)

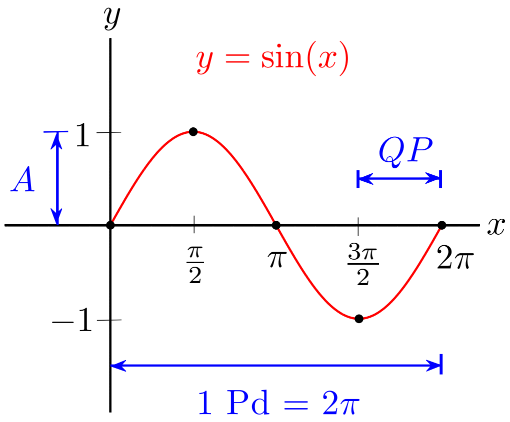
If $y = f(x)$ is a function and p is a nonzero constant such that $f(x) = f(x+p)$ for every x in the domain of f, then f is called a periodic function. The smallest such positive constant p is the period of the function.



The periodicity of the sine function is a result of the fact that co-terminal angles have the same sine.



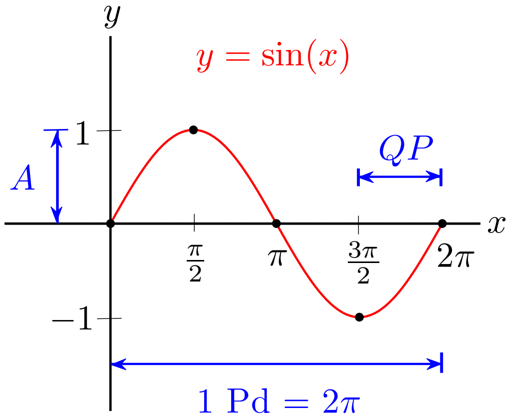
- The graph of $y = \sin(x)$ over any interval of 2π is called a one-period graph, the graph of one revolution, and/or one cycle of the sine wave.
- The graph of $y = \sin(x)$ over $[0, 2\pi]$ is called the fundamental cycle of $y = \sin(x)$.



Definition (Amplitude)

The amplitude of the sine function is

$$\text{Amplitude} = |A| = \frac{|max - min|}{2}$$



Definition (Quarter Point Width)

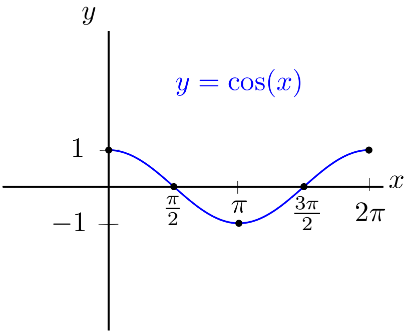
The Quarter Point Width (denoted QP) of the sine function is one period length divided by four.

$$QP = \frac{Pd}{4}$$

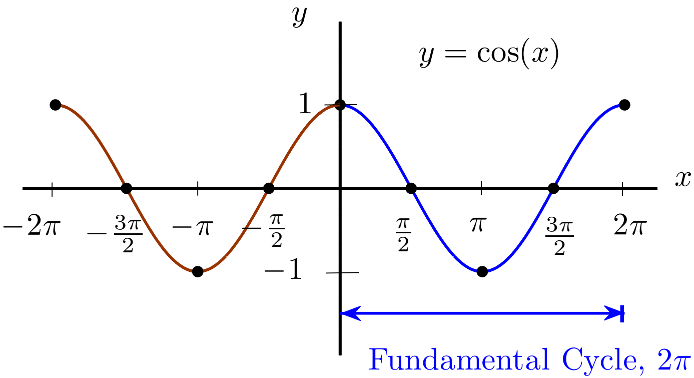
$f(x) = \cos(x)$

The graph of $y = \cos(x)$ also has five key points between 0 and 2π.

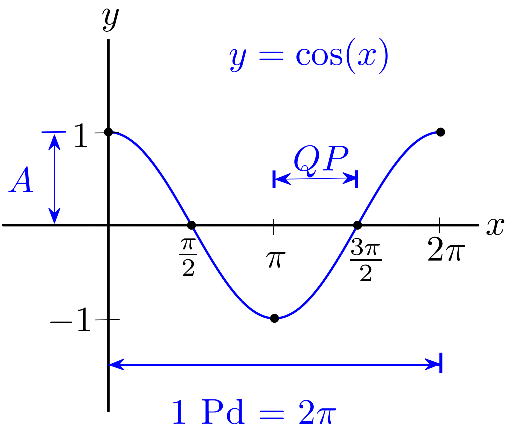
x	$f(x)$
0	1
$\frac{\pi}{2}$	0
π	-1
$\frac{3\pi}{2}$	0
2π	1



The five quarter points divide the x interval $[0, 2\pi]$ into *four* equal quarter point widths (QP).



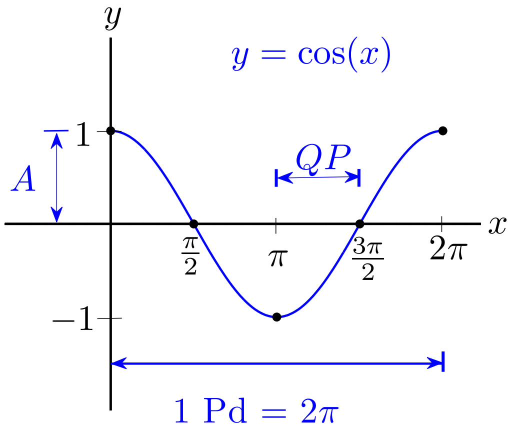
1. The cosine function is periodic, which means $cos(x + p) = cos(x)$ for every x, therefore copies of the graph (and the five key points) for $x \in [0, 2\pi]$ can be made to extend the graph over any domain.
2. The graph of $y = \cos(x)$ over $[0, 2\pi]$ is called the fundamental cycle of $y = \cos(x)$.



Definition (Amplitude)

The amplitude of the cosine function is

$$\text{Amplitude} = |A| = \frac{|max - min|}{2}$$



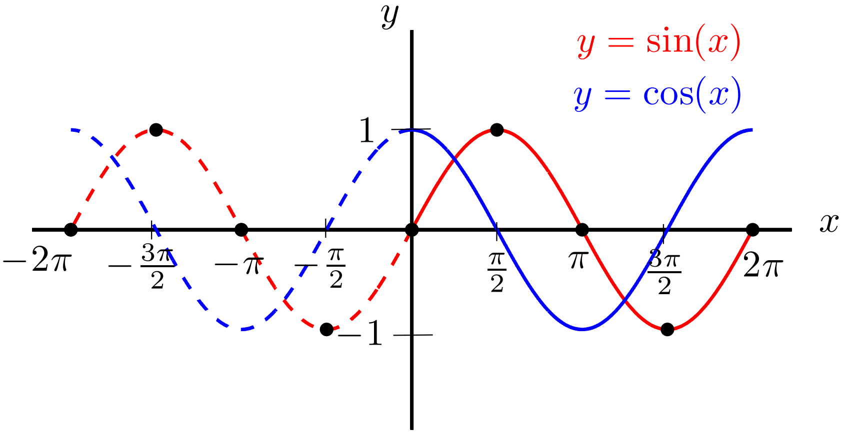
Definition (Quarter Point Width)

The Quarter Point Width (denoted QP) of the cosine function is one period length divided by four.

$$QP = \frac{Pd}{4}$$

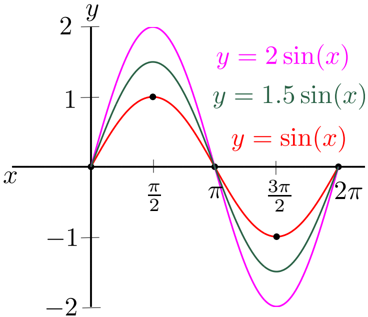
Both $f(x) = \sin(x)$ and $f(x) = \cos(x)$

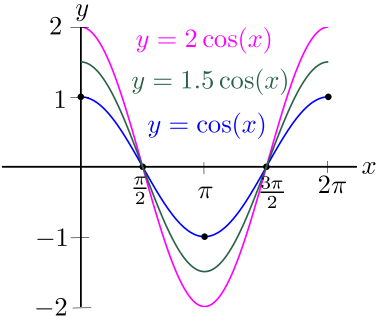
1. have domain $x \in (-\infty, \infty)$,
2. range $y \in [-1, 1]$,
3. fundamental period length, $Pd = 2\pi$,
4. Amplitude, $A = 1$.



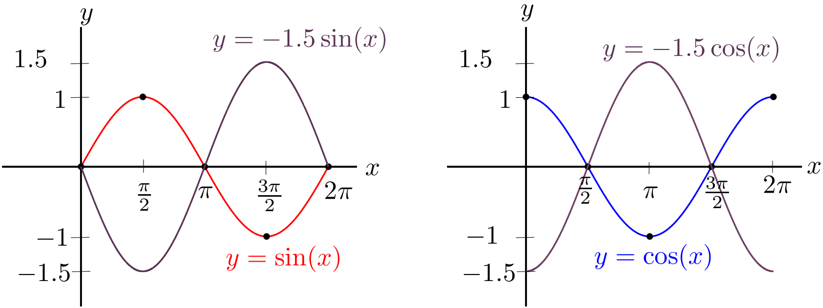
Theorem

The amplitude of $y = A \cdot \sin(x)$ *or* $y = A \cdot \cos(x)$ *is* $|A|$.





If the value of A is negative, a reflection and a magnification is applied to the graph of $y = \sin(x)$ or $y = \cos(x)$.



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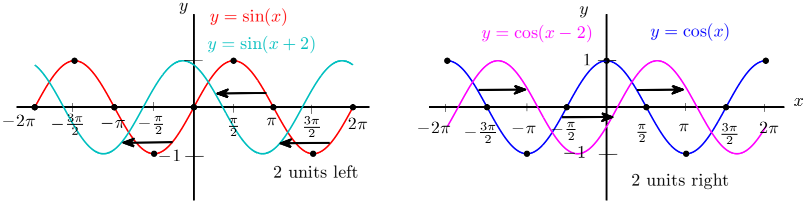
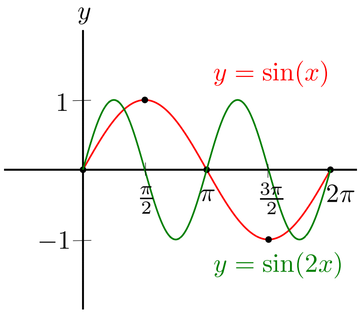
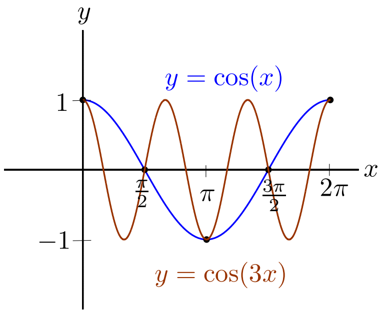


Definition (Phase Shift)

The phase shift, or horizontal shift of the graph of $y = \sin(x - C)$ or $y = \cos(x - C)$ is the number C. The shift is to the right if $C > 0$ and to the left if $C < 0$.

Example: For $y = \cos(x + \pi)$, is the shift to the right or left?



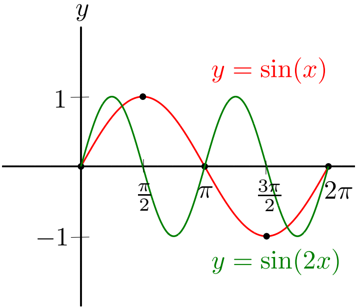


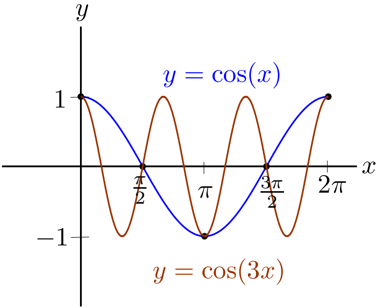
Definition (Changing the Period)

The period of the graph of $y = \sin(Bx)$ or

$y = \cos(Bx)$ is

$$Pd = \frac{2\pi}{B}.$$





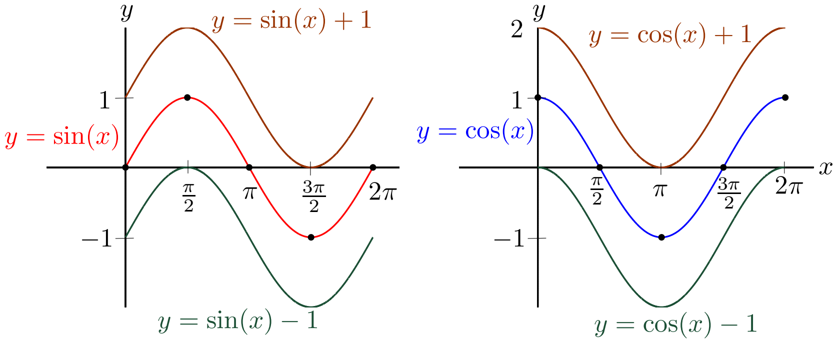
Definition (Frequency)

The frequency, F, of the graph of $y = \sin(Bx)$ or

$y = \cos(Bx)$ is

$$F = \frac{1}{Pd},$$

which represents the amount of time it takes to complete one cycle or revolution of the graph (if x represents the time axis).



Theorem (Vertical Translation)

The graphs of $y = \sin(x) + D$ and $y = \cos(x) + D$ are vertical translations of $y = \sin(x)$ and $y = \cos(x)$ a The vertical shift is down if $D < 0$ and up if $D > 0$.

Definitions

The generalized sine and cosine families of functions can be described by the two equations

$$f(x) = A\sin[B(x - C)] + D \quad \text{and} \quad f(x) = A\cos[B(x - C)] + D$$

where $A, B, C,$ and D are any real numbers.

- $|A|$ represents the <u>amplitude</u>, and amplitude$=|A| = \dfrac{|max - min|}{2}$
- The period is identified from $Pd = \dfrac{2\pi}{B}$
- C is the <u>phase shift</u>, or the <u>horizontal shift</u>.
- D is the amount of <u>vertical shift</u>.
- QP is the quarter-point width given by $QP = \dfrac{Pd}{4}$