**Tim Busken**
**The Binomial Distribution** **Name:** ____________________

Many probability experiments have only two outcomes. For example, when you guess at a multiple choice question, your answer is either right or wrong. A medical treatment can be considered effective or ineffective. When a coin is tossed it can land either heads or tails. Situations like these are called binomial experiments.

**Binomial experiments** have the following properties:

1. The procedure has a fixed number of trials.
2. The trials must be independent. (The outcome of any individual trial doesnt affect the probabilities in the other trials.)
3. Each trial must have only two possible outcomes (commonly referred to as success and failure).
4. The probability of a success remains the same in all trials.

**The outcomes of a binomial experiment and the corresponding probabilities of these outcomes are called a *binomial probability distribution*.**

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***Notation for Binomial Probability Distributions***:

$S$ and $F$ (success and failure) denote the two possible categories of all outcomes; $p$ and $q$ will denote the probabilities of $S$ and $F$, respectively.

| | |
|---|---|
| $P(S) = p$ | ($p$ =probability of success) |
| $P(F) = 1 - p = q$ | ($q$ =probability of failure) |
| $n$ | denotes the fixed number of trials. |
| $x$ | denotes a specific number of successes in $n$ trials, so $x$ can be any whole number between 0 and $n$, inclusive. |
| $p$ | denotes the probability of success in one of the $n$ trials. |
| $q$ | denotes the probability of failure in one of the $n$ trials. |
| $P(x)$ | denotes the probability of getting exactly $x$ successes among the $n$ trials. |

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***The Binomial Probability Formula***

$$P(x) = \left({}_nC_x\right) \cdot p^x \cdot q^{n-x}$$

for $x = 0, 1, 2, ..., n$, and recall that ${}_nC_x = \dfrac{n!}{(n-x)! \cdot x!}$.

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***Example***: Use the binomial probability formula to find the probability of $X$ successes if $n = 12$, $p = 0.75$ and $X = 5$. Show your work to get full credit.

1. ***Example***: A multiple choice test has 5 questions each of which has 5 possible answers, only one of which is correct. If Judy, who forgot to study for the test, guesses on all questions, what is the probability that she will answer exactly 2 questions correctly?

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**Solution:** The distribution of probabilities for the situation follows the Binomial Distribution, since the following is true.

✓ There are $n = 5$ trials (questions being answered).

✓ The trials are independent. The outcome from answering one question doesn't affect the probabilities associated with answering the other questions.

✓ Each trial (answering a question) has only two possible outcomes, with success being categorized as a correct answer.

✓ The probability for a success, $p = \frac{1}{5} = 0.2$, remains the same in all 10 trials.

We want to find $x = 2$ success from $n = 5$ trials with $p = 0.2$ and $q = 1 - p = 0.8$. Using the binomial formula, the probability of 2 success is

$$\begin{aligned} P(x=2) &= \left({}_nC_2\right) \cdot p^2 \cdot q^{n-2} \\ &= \left({}_5C_2\right) \cdot (0.2)^2 \cdot (0.8)^3 \\ &= \ 10 \ \cdot \ (0.2)^2 \cdot (0.8)^3 \\ &= \boxed{0.2048} \end{aligned}$$

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**Rationale for Using the Binomial Formula**

There are ${}_5C_2 = 10$ different ways to get two answers correct from five questions. The sample space for the binomial experiment contains pairs of numbers corresponding to *which* two questions Judy gets correct.

***Sample Space***

| | | | |
|---|---|---|---|
| #1, #2 | #2, #3 | #3, #4 | #4, #5 |
| #1, #3 | #2, #4 | #3, #5 | |
| #1, #4 | #2, #5 | | |
| #1, #5 | | | |

The probability is the same for any one of the pairs to occur and is equal to $(0.2)^2 \cdot (0.8)^3 = 0.02048$. For example, $P$(only questions #1 and #2 are correct) =

$= P$(#1 is correct AND #2 is correct AND #3 is wrong AND #4 is wrong AND #5 is wrong)

(Now use the Multiplication Rule for Independent Events.)

$= P(\#1 \text{ is correct}) \cdot P(\#2 \text{ is correct}) \cdot P(\#3 \text{ is wrong}) \cdot P(\#4 \text{ is wrong}) \cdot P(\#5 \text{ is wrong})$

$= 0.2 \cdot 0.2 \cdot 0.8 \cdot 0.8 \cdot 0.8$

$= (0.2)^2 \cdot (0.8)^3 = \boxed{0.02048}$

*Then* $P(\text{exactly 2 successes}) =$

$$= P\Bigg( \Big(\#1 \text{ AND } \#2 \text{ is correct}\Big) \text{ OR } \Big(\#1 \text{ AND } \#3 \text{ is correct}\Big) \text{ OR} \ldots$$

$$\Big(\#1 \text{ AND } \#4 \text{ is correct}\Big) \text{ OR } \Big(\#1 \text{ AND } \#5 \text{ is correct}\Big) \text{ OR } \ldots$$

$$\Big(\#2 \text{ AND } \#3 \text{ is correct}\Big) \text{ OR } \Big(\#2 \text{ AND } \#4 \text{ is correct}\Big) \text{ OR } \ldots$$

$$\Big(\#2 \text{ AND } \#5 \text{ is correct}\Big) \text{ OR } \Big(\#3 \text{ AND } \#4 \text{ is correct}\Big) \text{ OR } \ldots$$

$$\Big(\#3 \text{ AND } \#5 \text{ is correct}\Big) \text{ OR } \Big(\#4 \text{ AND } \#5 \text{ is correct}\Big) \Bigg)$$

Now use the Addition Rule. Since all ten events in the sample space are disjoint, this last probability can be written as this sum:

$$= P\Big(\#1 \text{ AND } \#2 \text{ is correct}\Big) + P\Big(\#1 \text{ AND } \#3 \text{ is correct}\Big) + \ldots$$

$$P\Big(\#1 \text{ AND } \#4 \text{ is correct}\Big) + P\Big(\#1 \text{ AND } \#5 \text{ is correct}\Big) + \ldots$$

$$P\Big(\#2 \text{ AND } \#3 \text{ is correct}\Big) + P\Big(\#2 \text{ AND } \#4 \text{ is correct}\Big) + \ldots$$

$$P\Big(\#2 \text{ AND } \#5 \text{ is correct}\Big) + P\Big(\#3 \text{ AND } \#4 \text{ is correct}\Big) + \ldots$$

$$P\Big(\#3 \text{ AND } \#5 \text{ is correct}\Big) + P\Big(\#4 \text{ AND } \#5 \text{ is correct}\Big)$$

$$\begin{aligned} = & (0.2)^2 \cdot (0.8)^3 + (0.2)^2 \cdot (0.8)^3 + \ldots \\ & (0.2)^2 \cdot (0.8)^3 + (0.2)^2 \cdot (0.8)^3 + \ldots \\ & (0.2)^2 \cdot (0.8)^3 + (0.2)^2 \cdot (0.8)^3 + \ldots \\ & (0.2)^2 \cdot (0.8)^3 + (0.2)^2 \cdot (0.8)^3 + \ldots \\ & (0.2)^2 \cdot (0.8)^3 + (0.2)^2 \cdot (0.8)^3 \end{aligned}$$

$$= \quad 10 \; \cdot \; (0.2)^2 \cdot (0.8)^3$$

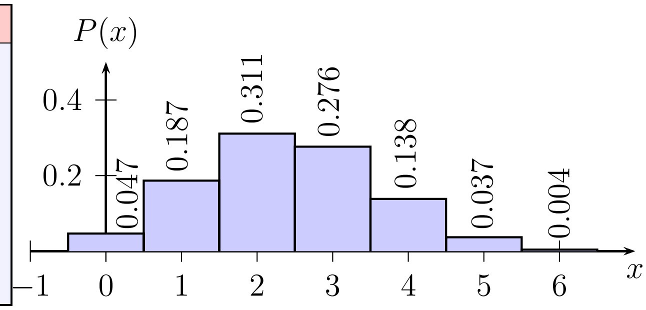
$$= \big({}_5C_2\big) \cdot (0.2)^2 \cdot (0.8)^3$$

$$= \big({}_nC_x\big) \cdot p^x \cdot q^{n-x}$$

***Example***: Assume that a procedure yields a binomial probability model with a trial repeated $n = 6$ times. Suppose the probability of success on a single trial is $p = 0.40$. Then, the probability model can be described with either the Binomial Formula, a table or a probability histogram.

$$P(X = x)$$
$$= \left({}_{n}C_{x}\right) \cdot p^{x} \cdot q^{n-x}$$
$$= \left({}_{6}C_{x}\right) \cdot (0.4)^{x} \cdot (0.6)^{6-x}$$

| $x$ | $P(X = x)$ |
|---|---|
| 0 | 0.047 |
| 1 | 0.187 |
| 2 | 0.311 |
| 3 | 0.276 |
| 4 | 0.138 |
| 5 | 0.037 |
| 6 | 0.004 |



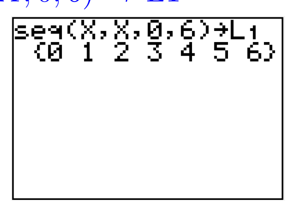

2. Use the model to find the probability of <u>exactly</u> $x = 3$ successes in $n = 6$ trials.

3. Find the probability of <u>at least</u> $x = 3$ successes in $n = 6$ trials.

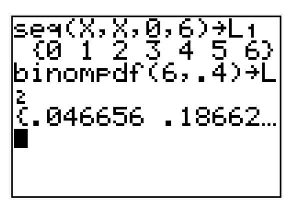
4. Find the probability of <u>at most</u> $x = 3$ successes in $n = 6$ trials.

5. <u>Calculator</u>: Place the model in your calculator's list 1 (L1) and list 2 (L2).

**<u>Step 1:</u>** Put the numbers 0 through 6 in L1. Use seq$(X, X, 0, 6) \rightarrow$ L1


**<u>Step 2:</u>** Use binomPdf$(n, p) \rightarrow$ L2 to put the associated probabilities in L2.


**<u>Step 3:</u>** Open your list environment (stat→enter) and notice the distribution table is in L1 and L2.

6. Assume that a procedure yields a binomial probability distribution with a trial repeated $n = 5$ times. Suppose the probability of success on a single trial is $p = 0.47$. Then $X$ counts the number of successes among 5 trials. Describe the probability distribution by filling out the table below. Round calculations to five decimal places. In addition, graph the distribution.

| $x$ | $P(X = x)$ |
|---|---|
| 0 | |
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |

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***More Calculator Facts — Binomial Model***

Your calculator has two binomial functions in its list of distributions (probability models). These are the $\boxed{\text{binomPdf}}$ and $\boxed{\text{binomCdf}}$ functions.

- pdf stands for <u>probability distribution function</u> and gives the probability $P(X = x)$
- cdf stands for <u>cumulative distribution function</u> and gives the probability $P(X \leq x)$

Both functions take can take 3 input arguments: $n, p$, and $x$, each separated by a comma.

| | TI–83/84 | Let $x = 3$ | TI–83/84 | |
|---|---|---|---|---|
| $P(X = x)$ | binomPdf$(n, p, x)$ | $P(X = 3)$ | binomPdf$(6, .4, 3)$ | 0.27648 |
| $P(X \leq x)$ | binomCdf$(n, p, x)$ | $P(X \leq 3)$ | binomCdf$(6, .4, 3)$ | 0.8208 |
| $P(X < x)$ | binomCdf$(n, p, x - 1)$ | $P(X < 3)$ | binomCdf$(6, .4, 2)$ | 0.54432 |
| $P(X > x)$ | $1 -$ binomCdf$(n, p, x)$ | $P(X > 3)$ | $1 -$ binomCdf$(6, .4, 3)$ | 0.1792 |
| $P(X \geq x)$ | $1 -$ binomCdf$(n, p, x - 1)$ | $P(X \geq 3)$ | $1 -$ binomCdf$(6, .4, 2)$ | 0.45568 |

7. Let $X$ be a binomial random variable with $n = 12$ and $p = 0.3$. Find the following:

   (a) $P(X = 5)$ (a) ____________

   (b) $P(X = 8)$ (b) ____________

(c) $P(X \leq 3)$ (c) ____________

(d) $P(X \leq 5)$ (d) ____________

(e) $P(X < 2)$ (e) ____________

(f) $P(X > 9)$ (f) ____________

(g) $P(X \geq 4)$ (g) ____________

(h) $P(3 \leq X \leq 8)$ (h) ____________

(i) $P(6 < X < 10)$ (i) ____________

8. Assume that 13% of people are left-handed. If we select 42 students at random, find the probability of each outcome described below. Use a binomial probability distribution.
   (a) There is at least one lefty in the group (a) ____________
   (b) There are exactly 3 lefties in the group (b) ____________
   (c) There are not more than 3 lefties in the group (c) ____________

9. People with type O-negative blood are said to be "universal donors." About 7% of the U.S. population has this blood type. Suppose that 335 people show up at a blood drive. Use a binomial probability distribution.
   (a) What is the expected number universal donors in the group?

   (a) ____________

   (b) What is the probability that exactly 30 universal donors are in the group?

   (b) ____________

   (c) What is the probability that more than 30 universal donors are in the group?

   (c) ____________

   (d) Using the range rule of thumb, would it be considered unusual to have that more than 50 universal donors are in the group?

   (d) ____________

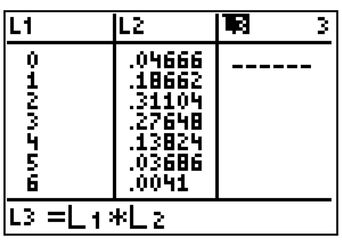
**Recall Question 6:** Assume that a procedure yields a binomial probability distribution with a trial repeated $n = 5$ times. Suppose the probability of success on a single trial is $p = 0.47$. Then $X$ counts the number of successes among 5 trials. Describe the probability distribution by filling out the table below. Round calculations to five decimal places.

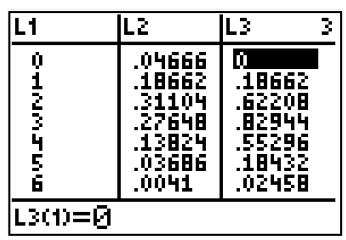
| $x$ | $P(X = x)$ |
|---|---|
| 0 | 0.04182 |
| 1 | 0.18543 |
| 2 | 0.32887 |
| 3 | 0.29164 |
| 4 | 0.12931 |
| 5 | 0.02293 |

10. Find the expected value, $\mu = E(X)$, of the above binomial distribution using the formula

$$E(X) = \sum x \cdot P(x)$$

Remember, this means multiplying the entries in L1 by the entries in L2, then summing the products.




11. Find the expected value of the distribution, $\mu = E(X)$, using the formula $E(X) = n \cdot p$. What do you notice?

12. Find the standard deviation of the binomial distribution, $\sigma$, using the formula $\sigma = \sqrt{n \cdot p \cdot q}$