1.3 Distance and Midpoint Formulas

Tim Busken

Graduate Teacher
Department of Mathematics
San Diego State University
Dynamical Systems Program

August 29, 2011

- In mathematics, a **theorem** is a statement that has been proven on the basis of previously established statements, such as other theorems, and previously accepted statements, such as axioms.

- In mathematics, a **theorem** is a statement that has been proven on the basis of previously established statements, such as other theorems, and previously accepted statements, such as axioms.
- $+, -, \times, \div$ are called the **arithmetic operators**.

- In mathematics, a **theorem** is a statement that has been proven on the basis of previously established statements, such as other theorems, and previously accepted statements, such as axioms.
- $+, -, \times, \div$ are called the **arithmetic operators**.
- When a letter represents any number from a set of numbers, it is called a **variable**.

- In mathematics, a **theorem** is a statement that has been proven on the basis of previously established statements, such as other theorems, and previously accepted statements, such as axioms.
- $+, -, \times, \div$ are called the **arithmetic operators**.
- When a letter represents any number from a set of numbers, it is called a **variable**.
- A **constant** is either a fixed number, such as 5, or a letter or symbol that represents a fixed number.

- An **algebraic expression** is any combination of variables, constants, grouping symbols, exponents and arithmetic operators.

- An **algebraic expression** is any combination of variables, constants, grouping symbols, exponents and arithmetic operators.
- In general, a **term** is a constant or the product of a constant and one or more variables raised to a power.

- An **algebraic expression** is any combination of variables, constants, grouping symbols, exponents and arithmetic operators.
- In general, a **term** is a constant or the product of a constant and one or more variables raised to a power. For example, consider the algebraic expression: $t + 29 - 5a^2b + 2x/y$.

- An **algebraic expression** is any combination of variables, constants, grouping symbols, exponents and arithmetic operators.
- In general, a **term** is a constant or the product of a constant and one or more variables raised to a power. For example, consider the algebraic expression:$t + 29 - 5a^2b + 2x/y$. The terms contained in the given expression are $t, 29, 5a^2b, and 2x/y$.

- An **algebraic expression** is any combination of variables, constants, grouping symbols, exponents and arithmetic operators.
- In general, a **term** is a constant or the product of a constant and one or more variables raised to a power. For example, consider the algebraic expression:$t + 29 - 5a^2b + 2x/y$. The terms contained in the given expression are $t, 29, 5a^2b, and 2x/y$.
- To **evaluate an algebraic expression**, substitute a numerical value for each variable into the expression and simplify the result by applying the order of operations in a left to right fashion.

- An **equation** is a statement (or sentence) indicating that two algebraic expressions are equal.

- An **equation** is a statement (or sentence) indicating that two algebraic expressions are equal.
- Suppose that a and b represent any real numbers, with the exception $a \neq 0$. A **linear equation in one variable** is an equation of the form $ax + b = 0$.

- An **equation** is a statement (or sentence) indicating that two algebraic expressions are equal.
- Suppose that a and b represent any real numbers, with the exception $a \neq 0$. A **linear equation in one variable** is an equation of the form $ax + b = 0$. For example, when $a = 1$ and $b = -4$, the resulting linear equation in x is $x - 4 = 0$.

- An **equation** is a statement (or sentence) indicating that two algebraic expressions are equal.
- Suppose that a and b represent any real numbers, with the exception $a \neq 0$. A **linear equation in one variable** is an equation of the form $ax + b = 0$. For example, when $a = 1$ and $b = -4$, the resulting linear equation in x is $x - 4 = 0$.
- If we replace the variable in a linear equation with a value that results in a true statement, we say that the value is a **solution** or **root** to the equation or that it **satisfies** the equation.

- An **equation** is a statement (or sentence) indicating that two algebraic expressions are equal.
- Suppose that a and b represent any real numbers, with the exception $a \neq 0$. A **linear equation in one variable** is an equation of the form $ax + b = 0$. For example, when $a = 1$ and $b = -4$, the resulting linear equation in x is $x - 4 = 0$.
- If we replace the variable in a linear equation with a value that results in a true statement, we say that the value is a **solution** or **root** to the equation or that it **satisfies** the equation.
- Ex: How do you solve $2x - 4 = 0$ for x?

- An **equation** is a statement (or sentence) indicating that two algebraic expressions are equal.
- Suppose that a and b represent any real numbers, with the exception $a \neq 0$. A **linear equation in one variable** is an equation of the form $ax + b = 0$. For example, when $a = 1$ and $b = -4$, the resulting linear equation in x is $x - 4 = 0$.
- If we replace the variable in a linear equation with a value that results in a true statement, we say that the value is a **solution** or **root** to the equation or that it **satisfies** the equation.
- Ex: How do you solve $2x - 4 = 0$ for x?

use the addn./subt. and mult./div. props of equality—Section 1.1

- An equation can be an classified as (1) an IDENTITY, otherwise it is classified as (2) CONDITIONAL, or (3) a CONTRADICTION.

- An equation can be an classified as (1) an IDENTITY, otherwise it is classified as (2) CONDITIONAL, or (3) a CONTRADICTION.
- An equation is called an **IDENTITY** when it is true for ALL VALUES of the variable for which the equation is defined.

- An equation can be an classified as (1) an IDENTITY, otherwise it is classified as (2) CONDITIONAL, or (3) a CONTRADICTION.
- An equation is called an **IDENTITY** when it is true for ALL VALUES of the variable for which the equation is defined. For example, the equation $x^2 - 49 = (x-7)(x+7)$ is an identity since it's a mathematical statement that is true for every real number x.

- An equation can be an classified as (1) an IDENTITY, otherwise it is classified as (2) CONDITIONAL, or (3) a CONTRADICTION.
- An equation is called an **IDENTITY** when it is true for ALL VALUES of the variable for which the equation is defined. For example, the equation $x^2 - 49 = (x-7)(x+7)$ is an identity since it's a mathematical statement that is true for every real number x.
- An equation is called **CONDITIONAL** when it is true for SOME VALUES of a variable, BUT NOT ALL.

- An equation can be an classified as (1) an IDENTITY, otherwise it is classified as (2) CONDITIONAL, or (3) a CONTRADICTION.
- An equation is called an **IDENTITY** when it is true for ALL VALUES of the variable for which the equation is defined. For example, the equation $x^2 - 49 = (x - 7)(x + 7)$ is an identity since it's a mathematical statement that is true for every real number x.
- An equation is called **CONDITIONAL** when it is true for SOME VALUES of a variable, BUT NOT ALL. For example $x^2 + 3x + 2 = 0$ is conditional since it is an equation that is only true for real numbers $x = -1, -2$

Equations involving Absolute Value

- Ex1: Solve $|x| = -2$ for x.
- Ex2: Solve $|x| = 4$ for x.
- Ex3: Solve $|x| = 0$ for x.

Equations involving Absolute Value

- Ex1: Solve $|x| = -2$ for x.
- Ex2: Solve $|x| = 4$ for x.
- Ex3: Solve $|x| = 0$ for x.

Theorem

Suppose k represent any positive real number. Then $|x| = k$ if and only if $x = \pm k$.

Equations involving Absolute Value

- Ex1: Solve $|x| = -2$ for x.
- Ex2: Solve $|x| = 4$ for x.
- Ex3: Solve $|x| = 0$ for x.

Theorem

Suppose k represent any positive real number. Then $|x| = k$ if and only if $x = \pm k$.

Comment: the variable x in the above theorem can represent any algebraic expression.

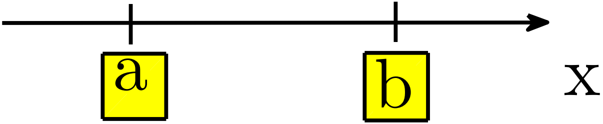
Ex: Solve $|x - 5| = 4$ for x.

The Cartesian Coordinate System

Read/view the definitions, graphs and corresponding discussion about the Cartesian Coordinate System on pages 102-103 of the text book.

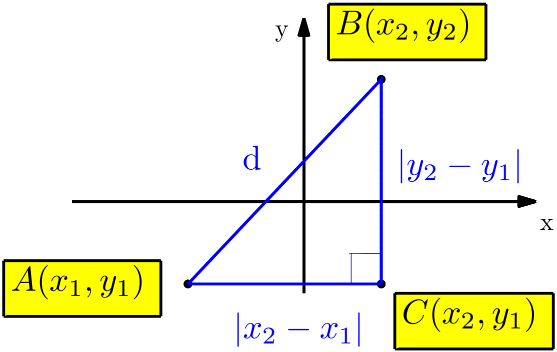
Theorem (Distance Formula: 1 dimension)

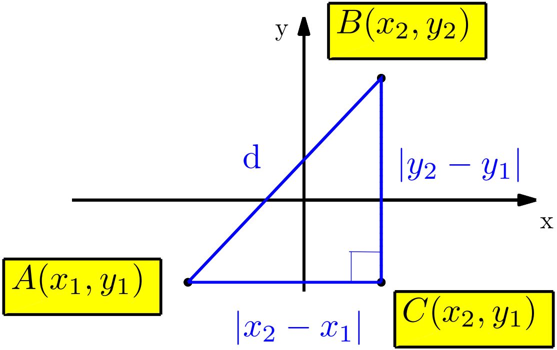
If a and b are real numbers, then the distance between them on a number line is $|a - b|$.



Distance Formula: 2 dimensions

Consider the points $A(x_1, y_1)$ and $B(x_2, y_2)$ in the figure below. Let d be the distance between points A and B (the HYPOTENUSE LENGTH of the right triangle). Since A and C lie on a horizontal line, the distance between them is $|x_2 - x_1|$. Likewise, $\overline{CB} = |y_2 - y_1|$.





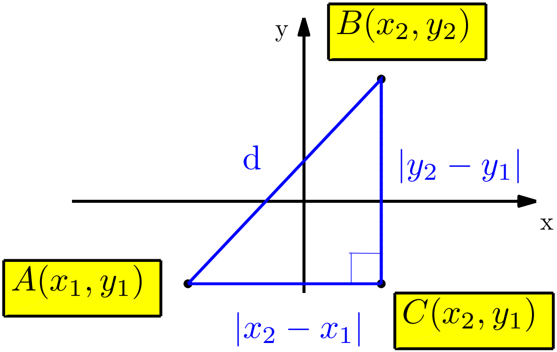
Since the sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse (Pyth. thm), then from the diagram

$$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$

Theorem (Distance Formula: 2 dimensions)

The distance d between the points (x_1, y_1) *and* (x_2, y_2) *is given by the formula*

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Example: Find the exact distance between the points $(5, -3)$ and $(-1, -6)$

Solution

Let $(x_1, y_1) = (5, -3)$ and $(x_2, y_2) = (-1, -6)$.

Then

$$
\begin{aligned}
d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{(-1 - 5)^2 + (-6 - (-3))^2} \\
&= \sqrt{(-6)^2 + (-3)^2} \\
&= \sqrt{36 + 9} = \sqrt{45} = 3\sqrt{5}
\end{aligned}
$$

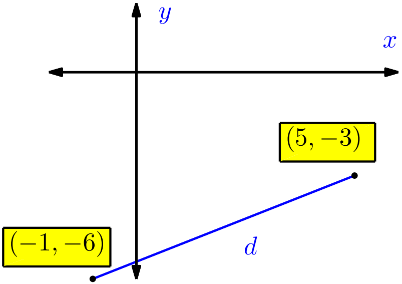
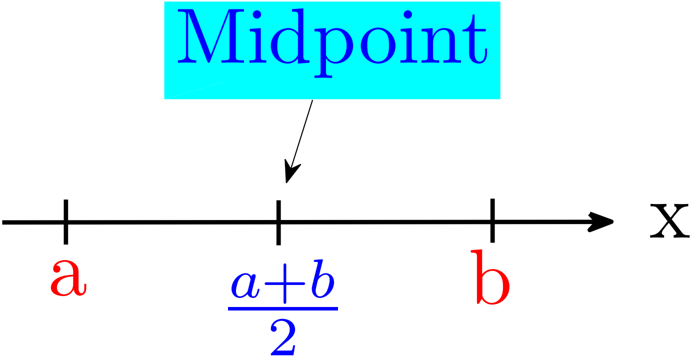


Figure:

Theorem (Midpoint Formula: 1 dimension)

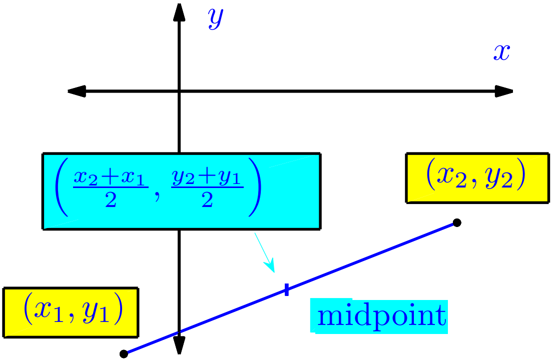
If a and b are real numbers, then the midpoint between them on a number line is $\frac{a+b}{2}$.



Theorem (Midpoint Formula: 2 dimensions)

Suppose (x_1, y_1) *and* (x_2, y_2) *are any two points in two-dimensional space. Then the midpoint of the line segment that joins them is:*

$$m = \left(\frac{(x_2 + x_1)}{2}, \frac{(y_2 + y_1)}{2}\right).$$



Example: Find the midpoint between the points $(5, -3)$ and $(-1, -6)$

Solution

Let $(x_1, y_1) = (5, -3)$ and $(x_2, y_2) = (-1, -6)$.
Then

$$m = \left(\frac{(x_2 + x_1)}{2}, \frac{(y_2 + y_1)}{2}\right)$$
$$= \left(\frac{5 + (-1)}{2}, \frac{-3 + (-6)}{2}\right)$$
$$= \left(\frac{4}{2}, \frac{-9}{2}\right)$$
$$= \left(2, -\frac{9}{2}\right)$$

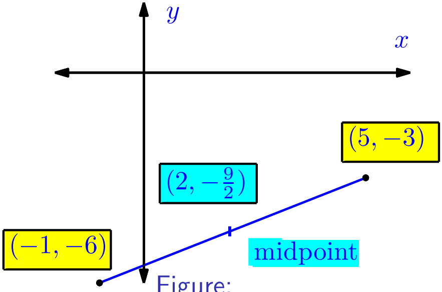


Figure:

- An **ordered pair** is a solution to an equation in two variables if the equation is correct when the variables are replaced by the coordinates of the ordered pair.

The Circle

- An **ordered pair** is a solution to an equation in two variables if the equation is correct when the variables are replaced by the coordinates of the ordered pair.
- The **solution set** to an equation in two variables is the set of all ordered pairs that satisfy the equation.

The Circle

- An **ordered pair** is a solution to an equation in two variables if the equation is correct when the variables are replaced by the coordinates of the ordered pair.
- The **solution set** to an equation in two variables is the set of all ordered pairs that satisfy the equation.
- The **graph of (the solution set to) an equation** in two variables is a two-dimensional geometric object that gives us a visual image of an algebraic object.

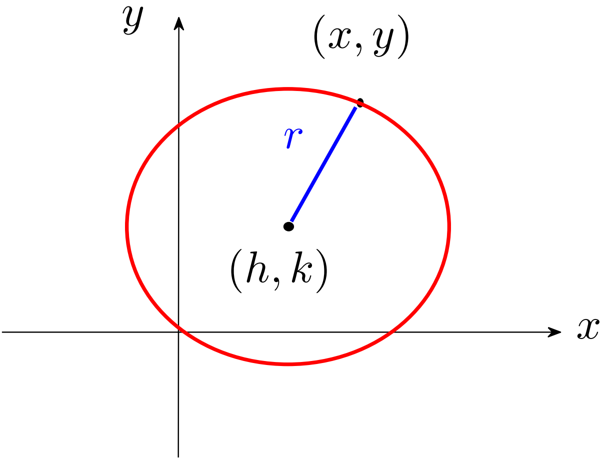
The Circle

- An **ordered pair** is a solution to an equation in two variables if the equation is correct when the variables are replaced by the coordinates of the ordered pair.
- The **solution set** to an equation in two variables is the set of all ordered pairs that satisfy the equation.
- The **graph of (the solution set to) an equation** in two variables is a two-dimensional geometric object that gives us a visual image of an algebraic object.

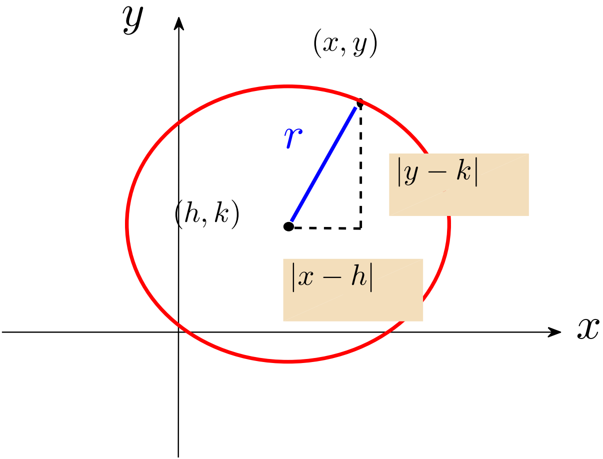
Definition (Circle)

A *circle* is defined by the set of all points in the xy plane that lie a fixed distance from a given point (the center). The fixed distance is called the *radius*, and the given point is the center.

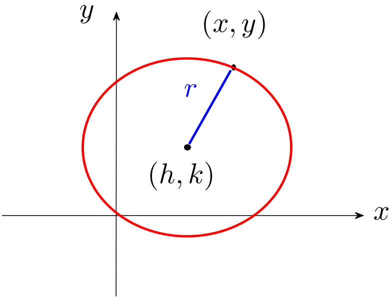
The distance formula can be used to write an equation for a circle with center (h, k) and radius r for $r > 0$.



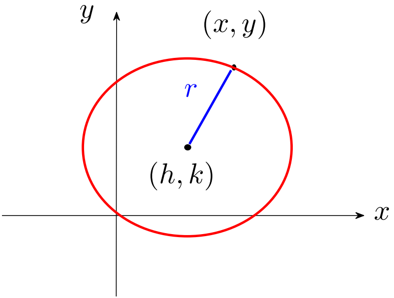
A point (x, y) is on the circle if and only if it satisfies the equation $\sqrt{(x-h)^2 + (y-k)^2} = r$.



Since both sides of the equation (previous slide) are positive, we can square each side to get the standard form for the equation of a circle.



Since both sides of the equation (previous slide) are positive, we can square each side to get the standard form for the equation of a circle.



Theorem (Equation for a Circle in Standard Form)

The equation for a circle with center (h, k) and radius r (where $r > 0$) is

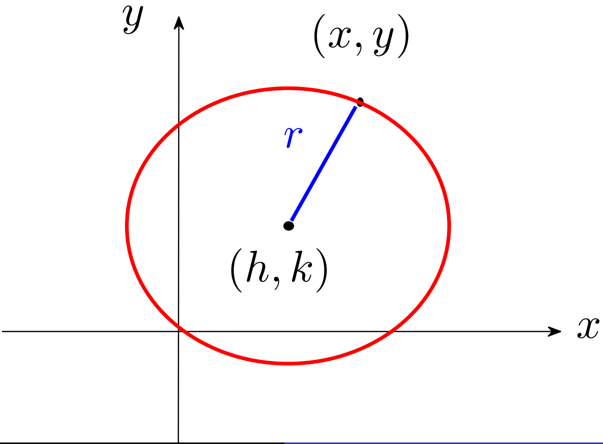
$$(x - h)^2 + (y - k)^2 = r^2$$

A circle centered at the origin has equation $x^2 + y^2 = r^2$.

Example: Sketch the graph of the equation

$$(x-1)^2 + (y+2)^2 = 3$$

.



Perfect Square Trinomials

Recall (chapter p.4) that algebraic expressions of the form $a^2 + 2ab + b^2$ have factorization $(a + b)^2$ and are called perfect square trinomials.

Perfect Square Trinomials

Recall (chapter p.4) that algebraic expressions of the form $a^2 + 2ab + b^2$ have factorization $(a + b)^2$ and are called perfect square trinomials. For instance,

$$x^2 + 10x + 25$$

is an example of a perfect square trinomial.

Perfect Square Trinomials

Recall (chapter p.4) that algebraic expressions of the form $a^2 + 2ab + b^2$ have factorization $(a + b)^2$ and are called perfect square trinomials. For instance,

$$x^2 + 10x + 25$$

is an example of a perfect square trinomial. The expression is called a trinomial because it is a polynomial (see chapter p.4) with three terms.

Perfect Square Trinomials

Recall (chapter p.4) that algebraic expressions of the form $a^2 + 2ab + b^2$ have factorization $(a + b)^2$ and are called perfect square trinomials. For instance,

$$x^2 + 10x + 25$$

is an example of a perfect square trinomial. The expression is called a trinomial because it is a polynomial (see chapter p.4) with three terms. But $x^2 + 10x + 25$ also has a perfect square factorization $(x + 5)^2$; hence the name (or classification) of $x^2 + 10x + 25$ as a perfect square trinomial.

Completing the Square

Completing the Square means *finding* the third term of a perfect square trinomial when given the first two.

Completing the Square

Completing the Square means *finding* the third term of a perfect square trinomial when given the first two. For example, consider the following algebraic expression with two terms:

$$x^2 + 10x.$$

Completing the Square

Completing the Square means *finding* the third term of a perfect square trinomial when given the first two. For example, consider the following algebraic expression with two terms:

$$x^2 + 10x.$$

If we add 25 to the given expression, well them we have completed the square since $x^2 + 10x + 25 = (x + 5)^2$.

Completing the Square means *finding* the third term of a perfect square trinomial when given the first two. For example, consider the following algebraic expression with two terms:

$$x^2 + 10x.$$

If we add 25 to the given expression, well them we have completed the square since $x^2 + 10x + 25 = (x + 5)^2$. The generalized rule for completing a square is as follows:

Theorem (Completing the square)

Let b be any real number. Given the first two terms of a quadratic expression $x^2 + bx$, the third term that must be added to the expression in order to render the expression a perfect square trinomial is $\left(\frac{b}{2}\right)^2$. That is,

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

In class examples: Webwork Homework Set 1.3, problems 1,4,5,6,8

Please print out and bring to class a pdf copy of your Webwork Homework Set 1.3