Key

**Math 150**
**Final Exam**
**Professor Busken** $y'(x) = 35x^6 + 24x^5 + 4x^3 + 6x^2 - 8x - 8$ **Name:** Key

1. (5 points) Let $y(x) = 5x^7 + 4x^6 + x^4 + 2x^3 - 4x^2 - 8x + 3$. Use the linearization of $y$ to approximate $y(2.0022)$

Let $x = 2$, $\Delta x = 0.0022$. The linear approximation theorem says

$f(x + \Delta x) \approx f(x) + f'(x)\,\Delta x$, or equivalently, that

$f(2.0022) \approx f(2) + f'(2)\cdot(0.0022) = 899 + 3040(0.0022)$

$= \boxed{905.688}$

2. (5 points) Find $\dfrac{dy}{dx}$ provided $y = 2^{\cot(x)}$.

$$\frac{d}{dx}\left(2^{\cot(x)}\right) = 2^{\cot(x)} \cdot \ln 2 \cdot \frac{d}{dx}\left(\cot(x)\right)$$

$$= 2^{\cot(x)} \ln 2 \cdot \left(-\csc^2(x)\right)$$

$$= \boxed{(-\ln 2) \cdot 2^{\cot(x)} \cdot \csc^2(x)}$$

3. (5 points) Find $\displaystyle\int_0^3 D_x\left[\sqrt{x^2+16}\right]\,dx$.

$$= \sqrt{x^2+16}\,\Big|_0^3 = \sqrt{25} - \sqrt{16} = 5 - 4 = 1$$

Key

4. (5 points) The radius r of a sphere is decreasing at a rate of 3 centimeters per second, how fast is the volume V changing when the radius is 2 centimeters long?

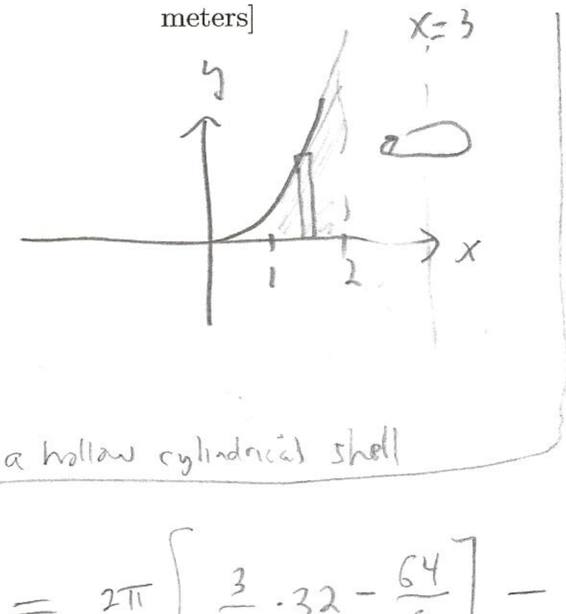
$\left[\text{Hint: } V_{\text{sphere}} = \frac{4}{3}\pi r^3\right]$

$$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$= 4\pi (2cm)^2 \cdot \left(-3 \frac{cm}{sec}\right)$$

$$= \boxed{-48\pi \frac{cm^3}{sec}} \approx -150 \frac{cm^3}{sec}$$

5. (5 points) Let $\Omega$ be the region bounded by: $y = x^4$ between $x = 1$ and $x = 2$. Find the volume of the solid obtained by revolving $\Omega$ about the line $l : x = 3$. [Distance in meters]




a hollow cylindrical shell

Shells

$$V = 2\pi \int_a^b (\text{avg radius})(\text{height})(\text{thickness } dx)$$

$$= 2\pi \int_1^2 \left[(3-x)x^4\right] dx = 2\pi \int_1^2 (3x^4 - x^5)\, dx$$

$$= 2\pi \left[\frac{3}{5}x^5 - \frac{x^6}{6}\right]_{x=1}^{x=2}$$

$$\boxed{\frac{81\pi}{5} m^3 \approx 50.89 m^3}$$

or $16.2\pi m^3$

$$= 2\pi \left[\frac{3}{5} \cdot 32 - \frac{64}{6}\right] - 2\pi \left[\frac{3}{5} - \frac{1}{6}\right] = 2\pi \left(\frac{96}{5} - \frac{32}{3} - \frac{3}{5} + \frac{1}{6}\right)$$

$$= 2\pi \left(\frac{93}{5} - \frac{64}{6} + \frac{1}{6}\right) = 2\pi \left(\frac{93}{5} - \frac{63}{6}\right) = 2\pi \left[\frac{93}{5} \cdot \frac{6}{6} - \frac{63}{6} \cdot \frac{5}{5}\right]$$

$$= 2\pi \left[\frac{558}{30} - \frac{315}{30}\right] = 2\pi \left[\frac{243}{30}\right] = \pi \left[\frac{243}{15}\right] = \pi \left[\frac{3 \cdot 81}{3 \cdot 5}\right] = \boxed{\frac{81\pi}{5} m^3}$$

6. (5 points) Find $y''$ given $y = \frac{1}{8}\int_1^{x^2} \frac{\sec^{-1}(x^2)}{\sqrt{x}}\,dx$

$$\frac{dy}{dx} = \frac{1}{8}\,\frac{\sec^{-1}(x^4)}{x}\cdot 2x = \frac{1}{4}\sec^{-1}(x^4)$$

$$\frac{d^2y}{dx^2} = \frac{1}{4}\cdot\frac{1}{x^4\sqrt{x^8-1}}\cdot\frac{d}{dx}(x^4)$$

$$= \frac{1}{4}\,\frac{4x^3}{x^4\sqrt{x^8-1}} = \frac{1}{x\sqrt{x^8-1}}$$

7. (5 points) The U.S. Postal Service will accept a package for domestic shipment only if the sum of its length and girth (distance around) does not exceed 108 inches. What are the dimensions of the largest cylindrical container that can be shipped?

Maximize Volume ($V = \pi r^2 h$) subject to the constraint


$h + 2\pi r \le 108$. From the contraint,

$$\boxed{\begin{aligned} r_{max} &= \frac{36}{\pi}\text{ in.} \\ h_{max} &= 36\text{ in} \end{aligned}}$$

$h = 108 - 2\pi r$, so that

$$V = V(r) = \pi r^2(108 - 2\pi r)$$
$$= 108\pi r^2 - 2\pi^2 r^3, \text{ and}$$

$\frac{dV}{dr} = 216\pi r - 6\pi^2 r^2 = 0$ whenever $6\pi r(36 - \pi r) = 0$. But $r \neq 0$, so $r = \frac{36}{\pi}$ is a possible extrema of $V(r)$. We do a 1st derivative test:

$$\begin{array}{ccccc} & + & & - & \\ & \nearrow & \rightarrow & \searrow & \\ 0 & & \frac{36}{\pi} & & \longrightarrow r,\ V'(r) \end{array}$$

$\exists$ a change in sign of $V'(r)$ around ch $x = \frac{36}{\pi}$. Moreover, $V''\left(\frac{36}{\pi}\right) = 216\pi - 12\pi^2\cdot\left(\frac{36}{\pi}\right)^2 > 0$

$\Rightarrow$ $r_{max} = \frac{36}{\pi}$ and $h_{max} = 108 - 2\pi r_{max} = 36.$

Key

8. (5 points) The interval $[0, 8]$ has been partitioned into $n$ subintervals and a Riemann Sum for $y = f(x)$ has been computed:

$$\left[\sum_{k=1}^{n} f(u_k)\,\Delta x_k = \sum_{k=1}^{n} \frac{1}{25 + u_k^2}\,\Delta x_k\right] \Rightarrow \left[f(x) = \frac{1}{25+x^2}\right]$$

Find $\int_0^5 f(x)dx$.

$$= \int_0^5 \frac{1}{25+x^2}\,dx = \left[\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right)\right]_{x=0}^{x=5} \quad \text{formula 17}$$

$$= \frac{1}{5}\tan^{-1}(1) - \frac{1}{5}\tan^{-1}(0) = \frac{1}{5}\frac{\pi}{4} - \frac{1}{5}\cdot 0 = \boxed{\frac{\pi}{20}} \approx 0.157$$

**For the next two questions, let** $\underset{\text{odd function}}{f(x)} = \frac{x^2-1}{x^3}$. $= x^{-3}(x^2-1) = x^{-1} - x^{-3}$

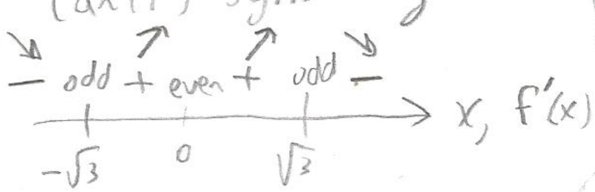
9. (5 points) Find (and identify) all **relative extrema** (if any) of $y = f(x)$.

$$f'(x) = -x^{-2} + 3x^{-4} = -x^{-4}(x^2-3) = -\frac{x^2-3}{x^4}$$

$$= -\frac{(x-\sqrt{3})(x+\sqrt{3})}{x^4} = 0 \text{ whenever } x = \pm\sqrt{3}.$$

Thus CN's of f are $x = \pm\sqrt{3}$ ($x=0$ not in dom$(f) \Rightarrow x=0$ not a CN).

Note that $f(x)$ is an odd function, thus its graph has origin (anti) symmetry.




min: $f(-\sqrt{3}) = -\frac{2}{9}\sqrt{3} \approx -0.3849$

max: $f(\sqrt{3}) = \frac{2}{9}\sqrt{3} \approx 0.3849$

The x-axis is a horizontal asymptote for f. Because $x = 0$ is an odd asymptote for f, $y \to \infty$ and $y \to -\infty$, so that the min $(-\sqrt{3}, f(-\sqrt{3}))$ and max $(\sqrt{3}, f(\sqrt{3}))$ are both relative and not absolute.

Key

10. (5 points) Find the intervals of concavity and all **points of inflection** (if any) of $y = f(x)$.

$$f''(x) = \frac{d}{dx}\left(-x^{-2}+3x^{-4}\right) = 2x^{-3}-12x^{-5} = 2x^{-5}\left[x^2-6\right] = \frac{2(x^2-6)}{x^5}$$

$$= \frac{2(x-\sqrt{6})(x+\sqrt{6})}{x^5}$$ so CN's of $f'$ are $x = \pm\sqrt{6}$. Thus,

$-$ odd $+$ odd $-$ odd $+$ $\rightarrow x, f''$

$-\sqrt{6}$ $0$ $\sqrt{6}$

note zero not in dom $f$) so its not a pt. of infl.

2nd deriv. test $\Longrightarrow$

$f$ is CU $\forall x \in (-\sqrt{6}, 0) \cup (\sqrt{6}, \infty)$

$f$ is CD $\forall x \in (-\infty, -\sqrt{6}) \cup (0, \sqrt{6})$

pts of inflection $(\pm\sqrt{6}, f(\pm\sqrt{6}))$

and $f(\pm\sqrt{6}) = \pm\frac{5\sqrt{6}}{36} \approx \pm 0.34$

11. (5 points) ***Extra Credit*** Let $f(x) = x^5+2x-1$. Use Newton's method to approximate to 8 decimal places the $x$-intercept of $y = f(x)$.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^5+2x_n-1}{5x_n^4+2}$$

let $x_0 = 1$

$x_1 \approx 0.714285714$

$x_2 \approx 0.528158734$

$x_3 \approx 0.487383194$

$x_4 \approx 0.486389536$

$x_5 \approx 0.486389036$

$x_6 \approx 0.486389036$

same

thus the root is $\boxed{x \approx 0.48638904}$ to 8 decimal places