Math 110 Review 4 Solns

Name: Key

No Calculators or Computing Devices allowed! Use Algebraic Notation AND Show All of Your Work.

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1. (6 points) Use Gaussian elimination to find the complete solution of the system, or show that no solution exists.

$\boxed{(x,y,z) = (5/2, 5/2, 0)}$

$-2\cdot \text{eqn}(1) + \text{eqn}(2) \to \text{eqn}(2)$

$$\begin{cases} x - y + 2z = 0 \\ 2x - 4y + 5z = -5 \\ 2y - 3z = 5 \end{cases}$$

$$= \left\{ \begin{array}{l} x - y + 2z = 0 \\ -2y + z = -5 \\ 2y - 3z = 5 \end{array} \right\} \text{eqn}2 + \text{eqn}3 \to \text{eqn}3 = \left\{ \begin{array}{l} x - y + 2z = 0 \\ -2y + z = -5 \\ -2z = 0 \end{array} \right\} \begin{array}{l} -\frac{1}{2}\,\text{eqn}\,2 \\ -\frac{1}{2}\,\text{eqn}(3) \end{array}$$

$$= \left\{ \begin{array}{l} x - y + 2z = 0 \\ y - \frac{1}{2}z = 5/2 \\ z = 0 \end{array} \right\} = \left\{ \begin{array}{l} x = 5/2 \\ y = 5/2 \\ z = 0 \end{array} \right\}$$

Check

(1) $x - y + 2z = 0$ ✓

(2) $2x - 4y + 5z = -5$

$2(5/2) - 4(5/2) = -5$

$5 - 10 = -5$ ✓

(3) $2y - 3z = 5$ ✓

2. (a) (2 points) Write a matrix equation equivalent to the following system.

$$\begin{cases} 4x - 3y = 10 \\ 3x - 2y = 30 \end{cases}$$

~~(a)~~ $\begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 30 \end{bmatrix}$

(b) (4 points) Find the inverse of the coefficient ~~matrix, and use it to~~ solve the system.

(b) $\boxed{(x,y) = (70,90)}$

$$A^{-1} = \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix} \quad \text{and}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 10 \\ 30 \end{bmatrix} = \begin{bmatrix} -2\cdot 10 + 3\cdot 30 \\ -3\cdot 10 + 4\cdot 30 \end{bmatrix} = \begin{bmatrix} -20+90 \\ -30+120 \end{bmatrix} = \begin{bmatrix} 70 \\ 90 \end{bmatrix}$$

$2\times 2 \quad 2\times 1$

Check: $4(70)-3(90) = 280-270 = 10$ ✓

$3(70) - 2(90) = 210-180 = 30$ ✓

3. (5 points) Solve $\left\{\begin{array}{l} x - y = 1 \\ 4x + 3y = 18 \end{array}\right\}$ using Cramer's Rule.

3. $\boxed{(x,y) = (3,2)}$

$$\begin{bmatrix} 1 & -1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 18 \end{bmatrix}; \quad x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}$$

$$D = \begin{vmatrix} 1 & -1 \\ 4 & 3 \end{vmatrix} = 3+4 = 7$$

$$D_x = \begin{vmatrix} 1 & -1 \\ 18 & 3 \end{vmatrix} = 3+18 = 21$$

$$D_y = \begin{vmatrix} 1 & 1 \\ 4 & 18 \end{vmatrix} = 18-4 = 14$$

$$x = \frac{D_x}{D} = \frac{21}{7} = 3$$

$$y = \frac{D_y}{D} = \frac{14}{7} = 2$$

4. Let $A = \begin{bmatrix} 1 & -5 \\ -3 & 7 \end{bmatrix}$, $\quad B = \begin{bmatrix} -2 & -6 \\ 2 & 7 \\ 1 & 0 \end{bmatrix}$, $\quad C = \begin{bmatrix} 1 & 3 & 1 \\ -2 & 7 & 2 \\ 0 & 2 & 4 \end{bmatrix}$

Carry out the indicated operation, or explain, using complete sentences, why it cannot be performed.

(a) (2 points) $A + B$ Can't be done! Matrix addition is only defined when both A and B have the same size (or order).

(b) (2 points) AB Can't be done! Matrix multiplication is only defined when the number of columns in A is equal to the number of rows in B.

A · B

2 x 2 3 x 2

cant do it

(c) (2 points) $BA - 3A$

$$BA - 3A = \begin{bmatrix} -2 & -6 \\ 2 & 7 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -5 \\ -3 & 7 \end{bmatrix} - 3 \begin{bmatrix} 1 & -5 \\ -3 & 7 \end{bmatrix}$$

B A

3 x 2 2 x 2

equal ✓

size

Can't be done since the size of BA (3 x 2) is not equal to the size of 3A (2 x 2). Subtraction of matrices is only defined when both matrices have the same size

(d) (2 points) B^{-1} Can't be done! For a matrix to be invertible, it must be square, and B is not a square matrix

(e) (2 points) $\det(B)$ Can't be done! Determinants are only for square matrices and B is not a square matrix

5. (6 points) Find the inverse of $C = \begin{bmatrix} 1 & 0 & 4 \\ -1 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}$ if it exists.

5. ________

$[A \vdots I_3] =$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right] R_1 + R_2 \to R_2$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 6 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right] -R_2 + R_3 \to R_3 = \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 6 & 1 & 1 & 0 \\ 0 & 0 & -3 & -1 & -1 & 1 \end{array}\right] -\frac{1}{3}R_3 \to R_3$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 6 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1/3 & 1/3 & -1/3 \end{array}\right] \begin{array}{l} -4R_3 + R_1 \to R_1 \\ -6R_3 + R_2 \to R_2 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/3 & -4/3 & 4/3 \\ 0 & 1 & 0 & -1 & -1 & 2 \\ 0 & 0 & 1 & 1/3 & 1/3 & -1/3 \end{array}\right] \Rightarrow \boxed{A^{-1} = \begin{bmatrix} -1/3 & -4/3 & 4/3 \\ -1 & -1 & 2 \\ 1/3 & 1/3 & -1/3 \end{bmatrix}}$$

check $A \cdot A^{-1} = I_3$

$$\begin{bmatrix} 1 & 0 & 4 \\ -1 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} -1/3 & -4/3 & 4/3 \\ -1 & -1 & 2 \\ 1/3 & 1/3 & -1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

6. (6 points) Find the partial fraction decomposition of $\frac{2x-3}{x^3+3x}$. $= \frac{2x-3}{x(x^2+3)}$

6. ________

$$\frac{2x-3}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$$

$$2x-3 = A(x^2+3) + (Bx+c)x$$

$$2x-3 = Ax^2+3A + Bx^2 + cx$$

$$2x-3 = (A+B)x^2 + Cx + 3A$$ now, equate the polynomial coefficients

$$\Rightarrow \left\{ \begin{array}{l} A+B=0 \\ c=2 \\ 3A=-3 \end{array} \right\} \equiv \left\{ \begin{array}{l} B=-A \\ C=2 \\ A=-1 \end{array} \right\} \equiv \left\{ \begin{array}{l} A=-1 \\ B=1 \\ C=2 \end{array} \right\}$$

Check $\quad \frac{-1}{x} + \frac{x+2}{x^2+3} = \frac{-(x^2+3)+x(x+2)}{x(x^2+3)}$

$$= \frac{-x^2-3+x^2+2x}{x(x^2+3)} = \frac{2x-3}{x^3+3x} \checkmark$$

So,

$$\boxed{\frac{2x-3}{x^3+3x} = -\frac{1}{x} + \frac{x+2}{x^2+3}}$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

7. Only one of the following two matrices has an inverse.

$$A = \begin{bmatrix} -2 & 5 & -2 \\ 0 & 7 & 0 \\ -2 & 1 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 5 & 6 \\ 2 & 7 & 1 \\ -7 & 1 & -3 \end{bmatrix}$$

$$\boxed{\begin{array}{l} \det(A) = 0 \\ \det(B) = 213 \end{array}}$$

(a) (5 points) Find the determinant of each matrix. (a)

$$\det(A) = 7 \begin{vmatrix} -2 & -2 \\ -2 & -2 \end{vmatrix} = 7\left[4 - 4\right] = \boxed{0}$$

$$\det(B) = 4 \begin{vmatrix} 7 & 1 \\ 1 & -3 \end{vmatrix} - 5 \begin{vmatrix} 2 & 1 \\ -7 & -3 \end{vmatrix} + 6 \begin{vmatrix} 2 & 7 \\ -7 & 1 \end{vmatrix}$$

$$= 4\left[-21 - 1\right] - 5\left[-6 + 7\right] + 6\left[2 + 49\right]$$

$$= 4(-22) - 5(1) + 6(51)$$

$$= -88 - 5 + 306 = -93 + 306 = \boxed{213}$$

$$\begin{array}{r} 306 \\ -93 \\ \hline 213 \end{array}$$

(b) (1 point) Use the determinants from part (a) to identify which matrix has an inverse.

A^{-1} doesn't exist since $\det(A) = 0$ (b) ________

B has an Inverse since $\det(B) \neq 0$

8. Let $A = \begin{bmatrix} 7 & -5 \\ -6 & 2 \\ -1 & 7 \end{bmatrix}$, $B = \begin{bmatrix} -2 & -6 \\ 2 & 7 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 3 & 1 \\ -2 & 7 & 2 \\ 0 & 2 & 4 \end{bmatrix}$

Carry out the indicated operation, or explain, using complete sentences, why it cannot be performed.

(a) (2 points) $A + B$ $= \begin{bmatrix} 7 & -5 \\ -6 & 2 \\ -1 & 7 \end{bmatrix} + \begin{bmatrix} -2 & -6 \\ 2 & 7 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7+(-2) & -5+(-6) \\ -6+2 & 2+7 \\ -1+0 & 7+1 \end{bmatrix}$

$= \begin{bmatrix} 5 & -11 \\ -4 & 9 \\ -1 & 8 \end{bmatrix}$

(b) (2 points) CA $\boxed{\text{Let } CA = E}$

$\underset{(3\times 3)}{} \cdot \underset{(3\times 2)}{} = \underset{(3\times 2)}{}$

$\begin{bmatrix} 1 & 3 & 1 \\ -2 & 7 & 2 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} 7 & -5 \\ -6 & 2 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \\ e_{31} & e_{32} \end{bmatrix} = \begin{bmatrix} -12 & 8 \\ -58 & 38 \\ -16 & 32 \end{bmatrix}$

$e_{11} = 1\cdot 7 + 3(-6) + 1(-1) = 7 - 18 - 1 = -12$

$e_{12} = 1\cdot(-5) + 3(2) + 1(7) = -5 + 6 + 7 = 8$

$e_{21} = -2\cdot 7 + 7(-6) + 2(-1) = -14 - 42 - 2 = -58$

$e_{22} = -2(-5) + 7(2) + 2(7) = 10 + 14 + 14 = 38$

$e_{31} = 0(7) + 2(-6) + 4(-1) = 0 + (-12) - 4 = -16$

$e_{32} = 0\cdot(-5) + 2(2) + 4(7)$
$= 4 + 28 = 32$

(c) (2 points) $2B - 3A$

$= 2\begin{bmatrix} -2 & -6 \\ 2 & 7 \\ 0 & 1 \end{bmatrix} + -3\begin{bmatrix} 7 & -5 \\ -6 & 2 \\ -1 & 7 \end{bmatrix}$

$= \begin{bmatrix} -4 & -12 \\ 4 & 14 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} -21 & 15 \\ 18 & -6 \\ 3 & -21 \end{bmatrix} = \begin{bmatrix} -25 & 3 \\ 22 & 8 \\ 3 & -19 \end{bmatrix}$

9. (6 points) Find the partial fraction decomposition of $\frac{3x-4}{x^3+4x^2}$.

$$\frac{3x-4}{x^2(x+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+4}$$

9. ________

$$3x-4 = Ax(x+4) + B(x+4) + Cx^2$$

$$3x-4 = Ax^2 + 4Ax + Bx + 4B + Cx^2$$

$$0x^2 + 3x - 4 = (A+C)x^2 + (4A+B)x + 4B$$

equate coefficients

$$\left\{\begin{array}{l} 0 = A + C \\ 3 = 4A + B \\ -4 = 4B \end{array}\right\} = \left\{\begin{array}{l} -C = A, \\ 3 = 4A + (-1) \\ -1 = B \end{array}\right\} = \left\{\begin{array}{l} C = -A \\ 4A = 4 \\ B = -1 \end{array}\right\}$$

$$= \left\{\begin{array}{l} A = 1 \\ B = -1 \\ C = -1 \end{array}\right\}$$

Check

$$\frac{1}{x} + \frac{-1}{x^2} + \frac{-1}{x+4} = \frac{x(x+4) - (x+4) - x^2}{x^2(x+4)}$$

$$= \frac{\cancel{x^2} + 4x - x - 4 - \cancel{x^2}}{x^2(x+4)} = \frac{3x-4}{x^2(x+4)} = \frac{3x-4}{x^3+4x^2} \checkmark$$

So, $\boxed{\frac{3x-4}{x^3+4x^2} = \frac{1}{x} + \frac{-1}{x^2} + \frac{-1}{x+4}}$

10. (6 points) Find the complete solution of the system, or show that no solution exists.

$$\begin{array}{l} ① \\ ② \\ ③ \end{array}\left\{\begin{aligned} x - 3y + 2z &= 12 \\ 2x - 5y + 5z &= 14 \\ x - 2y + 3z &= 20 \end{aligned}\right\}\begin{array}{l} \\ -2 \cdot \text{eqn}① + \text{eqn}② \to \text{eqn}② \\ -1\ \text{eqn}① + \text{eqn}③ \to \text{eqn}③ \end{array}$$

10.

$$= \left\{\begin{aligned} x - 3y + 2z &= 12 \\ y + z &= -10 \\ y + z &= 8 \end{aligned}\right\}$$

-2 eqn①	$-2x + 6y - 4z = -24$	
$+$ eqn②	$2x - 5y + 5z = 14$	
new eqn②	$0 + y + z = -10$	

Now replace eqn③ with $-1 \cdot$ eqn② + eqn③

-1 eqn①	$-x + 3y - 2z = -12$
$+$ eqn③	$x - 2y + 3z = 20$
new eqn③	$0 + y + z = 8$

$$= \left\{\begin{aligned} x - 3y + 2z &= 12 \\ y + z &= -10 \\ 0 &= 18 \end{aligned}\right\}$$

This last equation is false, so the system has $\boxed{\text{no solution}}$

11. (6 points) Find the complete solution of the system, or show that no solution exists.

Interchange ① and ③
$$\begin{cases} -3x - 5y + 36z = 10 \\ -x \quad\quad + 7z = 5 \\ x + y - 10z = -4 \end{cases}$$

11. ________

$$= \left\{\begin{array}{l} x + y - 10z = -4 \\ -x \quad + 7z = 5 \\ -3x - 5y + 36z = 10 \end{array}\right\}$$

eliminate the x term from equations ② and ③

$$\begin{array}{lr} \text{eqn ①} & x + y - 10z = -4 \\ + \text{eqn ②} & -x \quad + 7z = 5 \\ \hline \text{new eqn ②} & y - 3z = 1 \end{array}$$

$$= \left\{\begin{array}{l} x + y - 10z = -4 \\ y - 3z = 1 \\ -2y + 6z = -2 \end{array}\right\}$$

$$\begin{array}{lr} 3 \cdot \text{eqn ①} & 3x + 3y - 30z = -12 \\ + \text{eqn ③} & -3x - 5y + 36z = 10 \\ \hline \text{new eqn ③} & -2y + 6z = -2 \end{array}$$

Eliminate the y term from equation ③ now to put the system in reduced-echelon form.

$$\begin{array}{lr} 2 \cdot \text{eqn ②} & 2y - 6y = 2 \\ + \text{eqn ③} & -2y + 6y = -2 \\ \hline \text{new eqn. ③} & 0 + 0 = 0 \end{array}$$

Both variables were eliminated.

So, the system reduces to 2 eqns, namely $\left\{\begin{array}{l} x + y - 10z = -4 \\ y - 3z = 1 \end{array}\right\}$, or $\left\{\begin{array}{l} x = -y - 10z - 4 \\ y = 3z + 1 \end{array}\right\}$, and there are an infinite number of solutions. We let z take on any real number, so the soln set of ordered triples (x, y, z) can be written in terms of a single coordinate. Eqn ① in terms of z: $x = -y - 10z - 4$

$x = -3z + 1 - 10z - 4$

$\boxed{x = -13z - 5}$

Soln set $\{(x, y, z) = (-13z - 5,\ 3z + 1,\ z) \mid z \in \mathbb{R}\}$

12. (6 points) Sketch the graph (and label the vertices, or boundary intersections) of the solution set of ordered pairs of the system.

$$\begin{cases} x - y < 2 \\ x > 2 \\ y \leq 3 \end{cases}$$

$x - y = 2$

x-int $\quad x - 0 = 2$

$(2,0)$

y-int $\quad x - y = 2$

$0 - y = 2$

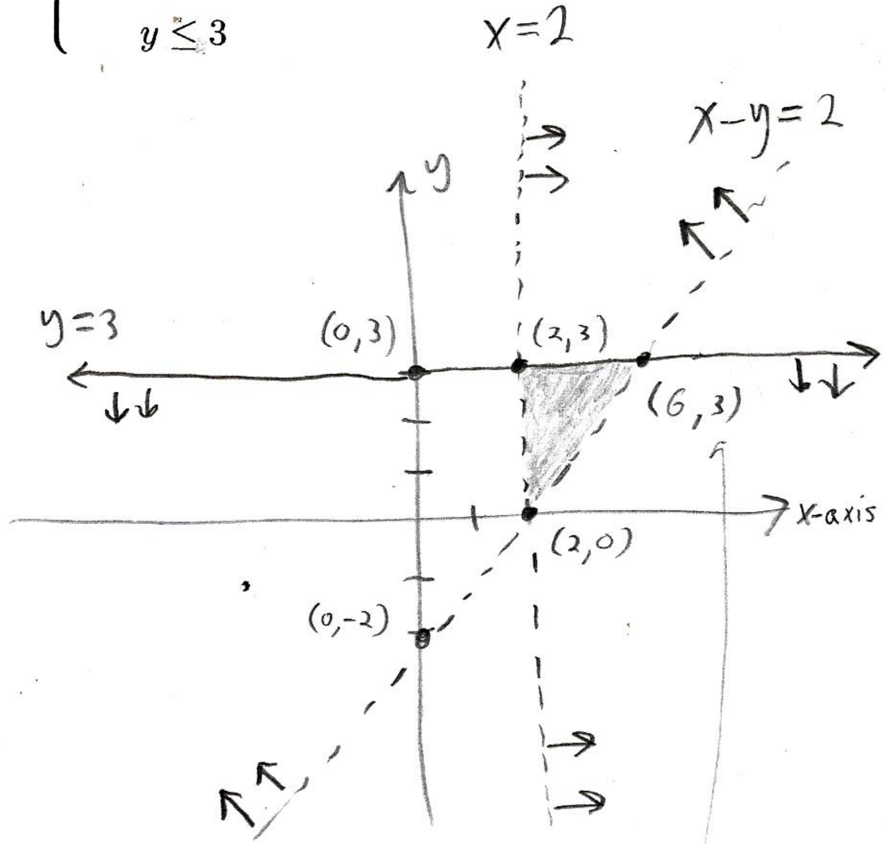
$y = -2$

$(0,-2)$

test pt $(x,y) = (0,0)$

$x - y < 2$

$0 - 0 < 2$ ✓



Intersection of $y=3$ with $x - y = 3$

When $x - 3 = 3$, or when $x = 6$

$\Rightarrow (6,3)$ is a vertex

The solution set of ordered pairs is the shaded region of the above graph