6.2 Division of Polynomials

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Dividing Polynomials

Simple Case: Division by a Monomial

Example: Divide $\dfrac{6x^3 - 9x^2 + 12x}{3x}$

Solution

$$\frac{6x^3 - 9x^2 + 12x}{3x} = \frac{1}{3x} \cdot (6x^3 - 9x^2 + 12x) = \frac{1}{3x} \cdot 6x^3 + \frac{1}{3x} \cdot (-9x^2) + \frac{1}{3x} \cdot 12x$$

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Solution

$$\frac{6x^3 - 9x^2 + 12x}{3x} = \frac{1}{3x} \cdot (6x^3 - 9x^2 + 12x) = \frac{1}{3x} \cdot 6x^3 + \frac{1}{3x} \cdot (-9x^2) + \frac{1}{3x} \cdot 12x$$

$$= \frac{6x^3}{3x} - \frac{9x^2}{3x} + \frac{12x}{3x} = 2x^2 - 3x + 4$$

Try This One! Divide $\dfrac{27x^4y^7 - 81x^5y^3}{-9x^3y^2}$ to lowest terms.

Procedure

Whenever the denominator is not a monomial, or a factor of the numerator, or if the numerator is not factorable, the previous method won't work. So, instead we use long division of polynomials, a method similar to long division of whole numbers.

Theorem (Division Algorithm:)

Suppose $D(x)$ and $P(x)$ are polynomial functions of x with $D(x) \neq 0$, and suppose that $D(x)$ is less than the degree of $P(x)$. Then there exist unique polynomials $Q(x)$ and $R(x)$, where $R(x)$ is either 0 or has degree less than the degree of $D(x)$, such that

$$P(x) = Q(x)D(x) + R(x) \quad \text{or, equivalently} \quad \frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

In words, we have

dividend = (quotient)(divisor) + remainder

or

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}.$$

Long Division of Polynomials

At any rate, $\dfrac{x^3+2x^2-x-2}{x-1}$ is equivalent to the following long division problem and solution:

$$
\begin{array}{rrrrrrrrl}
 & & x^2 & + & 3x & + & 2 & & \twoheadleftarrow \text{Quotient, Q(x)} \\
\cline{3-7}
\text{Divisor, D(x)} \twoheadrightarrow & x-1 & \big) \; x^3 & + & 2x^2 & - & x & -2 & \twoheadleftarrow \text{Dividend, P(x)} \\
 & & x^3 & - & x^2 & & & & \\
\cline{3-8}
 & & & & 3x^2 & - & x & -2 & \\
 & & & & 3x^2 & - & 3x & & \\
\cline{5-8}
 & & & & & & 2x & -2 & \\
 & & & & & & 2x & -2 & \\
\cline{7-8}
 & & & & & & & 0 & \twoheadleftarrow \text{Remainder, R(x)}
\end{array}
$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Step 1: set the problem up for long division:

$$x - 1 \overline{\smash{\big)}\, x^3 + 2x^2 - x - 2}$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Step 2: Compute $\frac{x^3}{x} = x^2$

$$\begin{array}{r|l} & x^2 \\ \hline x - 1 & x^3 + 2x^2 - x - 2 \end{array}$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Step 3: Compute the product $x^2 \cdot (x - 1)$ and list the result below $x^3 + 2x^2 - x - 2$

$$\begin{array}{r|l} & x^2 \\ \hline x - 1 & x^3 + 2x^2 - x - 2 \end{array}$$

Example: Divide $\dfrac{x^3+2x^2-x-2}{x-1}$

Solution: Step 3: Compute the product $x^2 \cdot (x-1) = x^3 - x^2$ and list the result below x^3+2x^2-x-2

$$\begin{array}{r|llll} & x^2 & & & \\ \cline{2-5} x-1 & x^3 + & 2x^2 - & x & -2 \\ & x^3 - & x^2 & & \\ \cline{2-5} \end{array}$$

Example: Divide $\dfrac{x^3+2x^2-x-2}{x-1}$

Solution: Step 4: CHANGE THE SIGNS AND ADD THE RESULT

$$
\begin{array}{r|llll}
 & x^2 & & & \\
\hline
x-1 & x^3 & +\ 2x^2 & -\ x & -\ 2 \\
 & -x^3 & +\ x^2 & & \\
\hline
 & 0 & +\ 3x^2 & & \\
\end{array}
$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Step 5: Bring $-x - 2$ down.

$$
\begin{array}{r|rrrr}
 & x^2 & & & \\
\cline{2-5}
x - 1 & x^3 & + 2x^2 & - x & - 2 \\
 & -x^3 & + x^2 & & \\
\cline{2-5}
 & & 3x^2 & - x & - 2
\end{array}
$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Repeat Steps 2 - 6 now on $3x^2 - x - 2$.

Step 2: Compute $\frac{3x^2}{x} = 3x$ and add the result to the quotient, $Q(x)$.

$$
\begin{array}{r|lllllll}
 & x^2 & + & 3x & & & & \\
\hline
x - 1 & x^3 & + & 2x^2 & - & x & - & 2 \\
 & -x^3 & + & x^2 & & & & \\
\hline
 & & & 3x^2 & - & x & - & 2
\end{array}
$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Repeat Step 3: Compute the product $3x \cdot (x - 1)$ and list the result below $3x^2 - x - 2$.

$$
\begin{array}{r|lllllll}
 & x^2 & + & 3x & & & & \\
\hline
(x-1) & x^3 & + & 2x^2 & - & x & - & 2 \\
 & -x^3 & + & x^2 & & & & \\
\hline
 & & & 3x^2 & - & x & - & 2
\end{array}
$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Repeat Step 3: Compute the product $3x \cdot (x - 1) = 3x^2 - 3x$
and list the result below $3x^2 - x - 2$.

$$
\begin{array}{r|rrrrrr}
 & x^2 & + & 3x & & & \\
\hline
(x-1) & x^3 & + & 2x^2 & - & x & -2 \\
 & -x^3 & + & x^2 & & & \\
\hline
 & & & 3x^2 & - & x & -2 \\
 & & & 3x^2 & - & 3x & \\
\hline
\end{array}
$$

Example: Divide $\dfrac{x^3+2x^2-x-2}{x-1}$

Solution: Repeat Step 4: CHANGE THE SIGNS AND ADD THE RESULT

$$
\begin{array}{r|llll}
 & x^2 & +\ 3x & & \\
\cline{2-5}
(x-1) & x^3 & +\ 2x^2 & -\ x & -\ 2 \\
 & -x^3 & +\ x^2 & & \\
\cline{2-5}
 & & 3x^2 & -\ x & -\ 2 \\
 & & {\color{red}-}\ 3x^2 & {\color{red}+}\ 3x & \\
\cline{3-5}
 & & 0 & +\ 2x & \\
\end{array}
$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Repeat Step 5: Bring -2 down.

$$
\begin{array}{r|lllllll}
 & x^2 & + & 3x & & & & \\
\hline
(x-1) & x^3 & + & 2x^2 & - & & x & -2 \\
 & -x^3 & + & x^2 & & & & \\
\hline
 & & & 3x^2 & & - & x & -2 \\
 & & & -3x^2 & & + & 3x & \\
\hline
 & & & 0 & & + & 2x & -2
\end{array}
$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Repeat Steps 2 - 6 now on $2x - 2$.

Step 2: Compute $\frac{2x}{x} = 2$ and add the result to the quotient, $Q(x)$.

$$
\begin{array}{r|rrrr}
 & x^2 & +\ 3x & +\ 2 & \\
\hline
(x-1) & x^3 & +\ 2x^2 & -\ x & -2 \\
 & -x^3 & +\ x^2 & & \\
\hline
 & & 3x^2 & -\ x & -2 \\
 & & -\ 3x^2 & +\ 3x & \\
\hline
 & & & 2x & -2
\end{array}
$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Repeat Step 3: Compute the product $2 \cdot (x - 1)$ and list the result below $2x - 2$.

$$
\begin{array}{r|rrrrrrr}
 & x^2 & + & 3x & + & & 2 & \\
\hline
(x-1) & x^3 & + & 2x^2 & - & & x & -2 \\
 & -x^3 & + & x^2 & & & & \\
\hline
 & & & 3x^2 & - & & x & -2 \\
 & & & -3x^2 & + & & 3x & \\
\hline
 & & & & & & 2x & -2
\end{array}
$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Repeat Step 3: Compute the product $2 \cdot (x - 1) = 2x - 2$ and list the result below $2x - 2$.

$$
\begin{array}{r|rrrr}
 & x^2 & +\ 3x & +\ 2 & \\
\hline
(x-1) & x^3 & +\ 2x^2 & -\ x & -2 \\
 & -x^3 & +\ x^2 & & \\
\hline
 & & 3x^2 & -\ x & -2 \\
 & & -\ 3x^2 & +\ 3x & \\
\hline
 & & & 2x & -2 \\
 & & & 2x & -\ 2 \\
\hline
\end{array}
$$

Example: Divide $\dfrac{x^3 + 2x^2 - x - 2}{x - 1}$

Solution: Repeat Step 4: CHANGE THE SIGNS AND ADD THE RESULT

$$
\begin{array}{r|rrrr}
 & x^2 & +\ 3x & +\ 2 & \\
\hline
(x-1) & x^3 + 2x^2 & -\ x & -2 & \\
 & -x^3 + x^2 & & & \\
\hline
 & 3x^2 & -\ x & -2 & \\
 & -\ 3x^2 & +\ 3x & & \\
\hline
 & & 2x & -2 & \\
 & & -2x & +2 & \\
\hline
 & & & 0 &
\end{array}
$$

Hence the remainder, $R(x) = 0$, and

$$\frac{x^3 + 2x^2 - x - 2}{x - 1} = x^2 + 3x + 2$$

How does one know when the long division process is finished?

It is not always the case that the remainder will be zero when dividing two polynomials. So, how does one know when the long division process of polynomials is over? When the degree of the remainder is less than the degree of the divisor.

In the example problem, the divisor was a first-degree polynomial. Following the guideline stated above, whenever the divisor is first-degree polynomial, we know the long division process is complete whenever the remainder is a constant (a polynomial with degree zero) or the number zero (a polynomial with no degree).

$$
\begin{array}{r|l}
 & x^2 + 3x + 2 \\
\cline{2-2}
x-1 & x^3 + 2x^2 - x - 2 \\
 & \underline{x^3 - x^2} \\
 & \quad\quad 3x^2 - x - 2 \\
 & \quad\quad \underline{3x^2 - 3x} \\
 & \quad\quad\quad\quad 2x - 2 \\
 & \quad\quad\quad\quad \underline{2x - 2} \\
 & \quad\quad\quad\quad\quad\quad 0
\end{array}
$$