# 7.1 Rational Exponents

Professor Tim Busken

Department of Mathematics
Grossmont College

October 29, 2012

**Definition**

The **square of a number** is the number times itself.

**Definition**

The **square of a number** is the number times itself.

**For instance**, the square of 4 is 16 because $4^2$ or $4 \cdot 4 = 16$. The square of $-4$ is also 16 because $(-4)^2 = (-4) \cdot (-4) = 16$.

**Definition**

The **square of a number** is the number times itself.

**For instance**, the square of 4 is 16 because $4^2$ or $4 \cdot 4 = 16$. The square of $-4$ is also 16 because $(-4)^2 = (-4) \cdot (-4) = 16$.

**Definition**

The reverse process of squaring is **finding a square root**.

**Definition**

The **square of a number** is the number times itself.

**For instance**, the square of 4 is 16 because $4^2$ or $4 \cdot 4 = 16$. The square of $-4$ is also 16 because $(-4)^2 = (-4) \cdot (-4) = 16$.

**Definition**

The reverse process of squaring is **finding a square root**.

**For example**, a square root of 16 is 4 because $4^2 = 16$. A square root of 16 is also $-4$ because $(-4)^2 = (-4) \cdot (-4) = 16$.

**Theorem**

*Every positive number has two square roots.*

## Theorem

*Every positive number has two square roots.*

**For instance**, the square roots of 25 are 5 and $-5$.

## Theorem

*Every positive number has two square roots.*

**For instance**, the square roots of 25 are 5 and –5.

## Definition

We use the symbol $\sqrt{\phantom{x}}$, called a **radical sign**, to indicate the positive square root.

**Theorem**

*Every positive number has two square roots.*

**For instance**, the square roots of 25 are 5 and –5.

**Definition**

We use the symbol $\sqrt{\phantom{x}}$, called a **radical sign**, to indicate the positive square root.

**For example**,

$$\sqrt{25} = 5 \text{ because } 5^2 = 25 \text{ and 5 is positive.}$$

$$\sqrt{9} = 3 \text{ because } 3^2 = 9 \text{ and 3 is positive.}$$

**Theorem**

*Every positive number has two square roots.*

**For instance**, the square roots of 25 are 5 and –5.

Note: it is a common mistake to assume that an expression like $\sqrt{25}$ indicates both square roots, 5 and –5. The expression $\sqrt{25}$ indicates only the positive square root of 25, which is 5. If we want the negative square root, we must use a negative sign in front of the radical sign.

**Theorem**

*Every positive number has two square roots.*

**For instance**, the square roots of 25 are 5 and –5.

Note: it is a common mistake to assume that an expression like $\sqrt{25}$ indicates both square roots, 5 and –5. The expression $\sqrt{25}$ indicates only the positive square root of 25, which is 5. If we want the negative square root, we must use a negative sign in front of the radical sign.

We write the negative square root of 25 as $-\sqrt{25}$ ( which is –5).

## Definition (Square Root of a Number)

The **square root**, $\sqrt{\ }$, of a positive number a is the positive number b whose square is a. In symbols,

$$\sqrt{a} = b \quad \text{if} \quad b^2 = a$$

For example,

$$\sqrt{36} = 6 \quad \text{if} \quad 6^2 = 36$$

**Find the square root of each.**

$\sqrt{100}$

$\sqrt{64}$

$-\sqrt{81}$

$-\sqrt{121}$

**Definition (Square Root of a Number)**

The **square root**, $\sqrt{\phantom{x}}$, of a positive number a is the positive number b whose square is a. In symbols,

$$\sqrt{a} = b \quad \text{if} \quad b^2 = a$$

For example,

$$\sqrt{36} = 6 \quad \text{if} \quad 6^2 = 36$$

**Find the square root of the following.**

$\sqrt{-16}$

**Definition (Square Root of a Number)**

The **square root**, $\sqrt{\phantom{x}}$, of a positive number a is the positive number b whose square is a. In symbols,

$$\sqrt{a} = b \quad \text{if} \quad b^2 = a$$

For example,

$$\sqrt{36} = 6 \quad \text{if} \quad 6^2 = 36$$

**Find the square root of the following.**

$\sqrt{-16}$

$\sqrt{-16}$ is not a real number since there is no real number we can raise to the second power and obtain $-16$.

## Definition (Square Root of a Number)

The **square root**, $\sqrt{\ }$, of a positive number a is the positive number b whose square is a. In symbols,

$$\sqrt{a} = b \quad \text{if} \quad b^2 = a$$

For example,

$$\sqrt{36} = 6 \quad \text{if} \quad 6^2 = 36$$

**Find the square root of each.**

$$\sqrt{\frac{1}{4}}$$

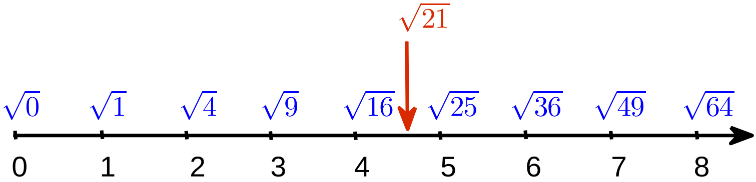
$$-\sqrt{\frac{49}{16}}$$

$$-\sqrt{\frac{4}{25}}$$

## Definition

Numbers like $\frac{1}{4}, \frac{4}{25}$, 9 and 36 are called **perfect squares** because their square root is a whole number or a fraction.

A square root such as $\sqrt{21}$ cannot be written as a whole number or a fraction since 21 is not a perfect square. It can be approximated by estimating, by using a table, or by using a calculator. We can however, estimate what two whole numbers $\sqrt{21}$ is between.



## Definition (Cube Root of a Number)

The **cube root**, $\sqrt[3]{\phantom{a}}$, of a number a is the number b whose cube is a. In symbols,

$$\sqrt[3]{a} = b \quad \text{if} \quad b^3 = a$$

For example,

$$\sqrt[3]{27} = 3 \quad \text{since} \quad 3^3 = 27$$

**Find the cube root of each.**

$\sqrt[3]{8}$

$\sqrt[3]{-8}$

$-\sqrt[3]{\frac{1}{8}}$

## Definition (Cube Root of a Number)

The **cube root**, $\sqrt[3]{\phantom{x}}$, of a number a is the number b whose cube is a. In symbols,

$$\sqrt[3]{a} = b \quad \text{if} \quad b^3 = a$$

For example,

$$\sqrt[3]{27} = 3 \quad \text{since} \quad 3^3 = 27$$

## Definition

An expression like $-\sqrt[3]{\frac{1}{8}}$ involving a radical sign is called a **radical expression**. In the radical expression $-\sqrt[3]{\frac{1}{8}}$, the number 3 is called the **index** of the radical, and $\frac{1}{8}$ is called the **radicand**.

## Definition (Cube Root of a Number)

The **cube root**, $\sqrt[3]{\phantom{a}}$, of a number a is the number b whose cube is a. In symbols,

$$\sqrt[3]{a} = b \quad \text{if} \quad b^3 = a$$

For example,

$$\sqrt[3]{27} = 3 \quad \text{since} \quad 3^3 = 27$$

## Definition (More Fine Print)

The index of a radical must be a positive integer greater than 1. If no index is written, it is assumed to be 2.

**Definition (Fourth Root of a Number)**

The **fourth root**, $\sqrt[4]{\ }$, of a positive number a is the number b such that

$$\sqrt[4]{a} = b \quad \text{if} \quad b^4 = a$$

For example,

$$\sqrt[4]{16} = 2 \quad \text{since} \quad 2^4 = 16$$

**Find the fourth root of each.**

$\sqrt[4]{1}$

$-\sqrt[4]{\frac{1}{16}}$

$\sqrt[4]{-16}$

## Definition (Fourth Root of a Number)

The **fourth root**, $\sqrt[4]{\phantom{x}}$, of a positive number a is the number b such that

$$\sqrt[4]{a} = b \quad \text{if} \quad b^4 = a$$

For example,

$$\sqrt[4]{16} = 2 \quad \text{since} \quad 2^4 = 16$$

**Find the fourth root of each.**

$\sqrt[4]{1}$

$-\sqrt[4]{\frac{1}{16}}$

$\sqrt[4]{-16}$

$\sqrt[4]{-16}$ is not a real number since there is no real number we can raise to the fourth power and obtain $-16$.

There are also fifth roots, sixth roots, seventh roots, and so on. As a generalization, we call $\sqrt[n]{a}$ the $n^{th}$ root of $a$.

There are also fifth roots, sixth roots, seventh roots, and so on. As a generalization, we call $\sqrt[n]{a}$ the $n^{th}$ root of $a$.

**Definition (The $n^{th}$ Root of a Number)**

The **$n^{th}$ root**, $\sqrt[n]{a}$, of a positive number a is the number b such that

$$\sqrt[n]{a} = b \quad \text{if} \quad b^n = a$$

There are also fifth roots, sixth roots, seventh roots, and so on. As a generalization, we call $\sqrt[n]{a}$ the $n^{th}$ root of $a$.

**Definition (The $n^{th}$ Root of a Number)**

The $n^{th}$ **root**, $\sqrt[n]{a}$, of a positive number a is the number b such that

$$\sqrt[n]{a} = b \quad \text{if} \quad b^n = a$$

We can use the following chart to help summarize the fine print of the definition.

| $n$ | $a$ | $\sqrt[n]{a}$ | $\sqrt[n]{a^n}$ |
|---|---|---|---|
| Even | Positive | Positive | $a$ |
| | Negative | Not a real number | $-a$ |
| Odd | Positive | Positive | $a$ |
| | Negative | Negative | $a$ |

There are also fifth roots, sixth roots, seventh roots, and so on. As a generalization, we call $\sqrt[n]{a}$ the $n^{th}$ root of $a$.

**Definition (The $n^{th}$ Root of a Number)**

The $n^{th}$ **root**, $\sqrt[n]{a}$, of a positive number a is the number b such that

$$\sqrt[n]{a} = b \quad \text{if} \quad b^n = a$$

We can use the following chart to help summarize the fine print of the definition.

| $n$ | $a$ | $\sqrt[n]{a}$ | $\sqrt[n]{a^n}$ |
|---|---|---|---|
| Even | Positive | Positive | $a$ |
| | Negative | Not a real number | $-a$ |
| Odd | Positive | Positive | $a$ |
| | Negative | Negative | $a$ |

It needs to be clear that we cannot take an even root of a negative number!!!

# Rational Numbers as Exponents

In this class, whenever we encounter variables underneath the radical sign, we assume all variables represent nonnegative numbers.

In this class, whenever we encounter variables underneath the radical sign, we assume all variables represent nonnegative numbers.

Simplify each radical expression as much as possible. Assume all variables represent nonnegative numbers.

$\sqrt{4x^2}$

$\sqrt[3]{-8x^6y^9}$

$\sqrt[4]{\dfrac{b^{12}}{16}}$

$\sqrt[5]{-32m^5}$

# Rational Numbers as Exponents

We have encountered exponential expressions like $2^3$ and $(-2x)^5$ which have integers exponents. But what about expressions like $2^{1/2}$ and $(3x)^{3/5}$ which have integers exponents?

# Rational Numbers as Exponents

We have encountered exponential expressions like $2^3$ and $(-2x)^5$ which have integers exponents. But what about expressions like $2^{1/2}$ and $(3x)^{3/5}$ which have integers exponents?

**Definition (The Fractional Exponent Rule)**

Suppose $d$ is a positive integer and suppose $a$ is a real number. Then

$$\sqrt[d]{a} = a^{1/d}$$

(but $a$ must not be negative when the index is even).

# Rational Numbers as Exponents

We have encountered exponential expressions like $2^3$ and $(-2x)^5$ which have integers exponents. But what about expressions like $2^{1/2}$ and $(3x)^{3/5}$ which have integers exponents?

**Definition (The Fractional Exponent Rule)**

Suppose $d$ is a positive integer and suppose $a$ is a real number. Then

$$\sqrt[d]{a} = a^{1/d}$$

(but $a$ must not be negative when the index is even).

# Rational Numbers as Exponents

We have encountered exponential expressions like $2^3$ and $(-2x)^5$ which have integers exponents. But what about expressions like $2^{1/2}$ and $(3x)^{3/5}$ which have integers exponents?

**Definition (The Fractional Exponent Rule)**

Suppose $d$ is a positive integer and suppose $a$ is a real number. Then

$$\sqrt[d]{a} = a^{1/d}$$

(but $a$ must not be negative when the index is even).

For example, we can rewrite $\sqrt{16} = 16^{1/2}$.

# Rational Numbers as Exponents

The next theorem can be proved using properties of exponents.

The next theorem can be proved using properties of exponents.

**Theorem (The Fractional Exponent Rule)**

*Suppose $n$ and $d$ are positive integers and suppose $a$ is a real number. Then*

$$\sqrt[d]{a^n} = a^{n/d}$$

*(but $a$ must not be negative when the index is even).*

The next theorem can be proved using properties of exponents.

**Theorem (The Fractional Exponent Rule)**

*Suppose $n$ and $d$ are positive integers and suppose $a$ is a real number. Then*

$$\sqrt[d]{a^n} = a^{n/d}$$

*(but $a$ must not be negative when the index is even).*

The fractional exponent rule can be used as the mnemonic "dan becomes and," or "I can remember the dan and rule," etc.

# Rational Numbers as Exponents

The next theorem can be proved using properties of exponents.

**Theorem (The Fractional Exponent Rule)**

*Suppose $n$ and $d$ are positive integers and suppose $a$ is a real number. Then*

$$\sqrt[d]{a^n} = a^{n/d}$$

*(but $a$ must not be negative when the index is even).*

The fractional exponent rule can be used as the mnemonic "dan becomes and," or "I can remember the dan and rule," etc.

## Theorem (The Fractional Exponent Rule)

$$\sqrt[d]{a^n} = a^{n/d}$$

Write each expression as a radical expression and then simplify the result, if possible.

$(-8)^{1/3}$

$-(144)^{1/2}$

$(-144)^{1/2}$

$\left(-144^{\frac{1}{2}}\right)$

$-(81)^{1/4}$

**Theorem (The Fractional Exponent Rule)**

$$\sqrt[d]{a^n} = a^{n/d}$$

Write the radical expression with a rational exponent and then simplify the result, if possible.

$$\sqrt[4]{x^4 y^8}$$

$$\sqrt[3]{x^6 y^{18}}$$

$$\sqrt[2]{\frac{25x^2}{36}}$$

**Theorem (The Fractional Exponent Rule)**

$$a^{\frac{n}{d}} = \left(a^{\frac{1}{d}}\right)^n = (a^n)^{\frac{1}{d}}$$

Simplify as much as possible.

$9^{3/2}$

$16^{3/4}$

$8^{-2/3}$

$\left(\frac{16}{81}\right)^{-3/4}$

## Theorem (The Fractional Exponent Rule)

$$a^{\frac{n}{d}} = \left(a^{\frac{1}{d}}\right)^n = (a^n)^{\frac{1}{d}}$$

Assume the variables represent positive quantities and simplify as much as possible.

$x^{\frac{1}{3}} \cdot x^{\frac{5}{3}}$

$y^{-3/8} \cdot y^{5/12} \cdot y^{7/9}$

$(x^{2/3})^{3/4}$

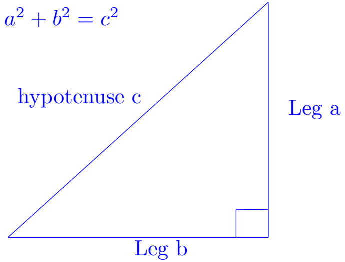
$\dfrac{x^{3/4}}{x^{2/3}}$

$\dfrac{(x^{1/3}y^{-3})^6}{x^4y^{10}}$

# The Pythagorean Theorem and Square Roots

**Theorem**

*If a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse, then* $a^2 + b^2 = c^2$.




The Pythagorean Equation, $c^2 = a^2 + b^2$, can be written as $c = \sqrt{a^2 + b^2}$